Problem 3.8

(a) Check that the eigenvalues of the hermitian operator in Example 3.1 are real. Show that the eigenfunctions (for distinct eigenvalues) are orthogonal.

(b) Do the same for the operator in Problem 3.6.

Solution

Part (a)

The hermitian operator in Example 3.1 is

$$i\frac{d}{d\phi},$$

and its eigenfunctions are $e^{-in\phi}$, where n is an integer and $0 \leq \phi \leq 2\pi$.

$$i\frac{d}{d\phi}(e^{-in\phi}) = i(-in)(e^{-in\phi}) = n(e^{-in\phi})$$

The eigenvalue corresponding to $e^{-in\phi}$ is n, which is real. To show that the eigenfunctions are orthogonal, take the inner product of $e^{-in_1\phi}$ and $e^{-in_2\phi}$ (with $n_1 \neq n_2$) and show that it's zero.

$$\begin{aligned}
\int_0^{2\pi} (e^{-in_1\phi})^*(e^{-in_2\phi})\,d\phi &= \int_0^{2\pi} (e^{in_1\phi})(e^{-in_2\phi})\,d\phi \\
&= \int_0^{2\pi} e^{i(n_1-n_2)\phi}\,d\phi \\
&= \frac{1}{i(n_1-n_2)} e^{i(n_1-n_2)\phi}\Big|_0^{2\pi} \\
&= \frac{1}{i(n_1-n_2)}(e^{2\pi i(n_1-n_2)} - e^0) \\
&= \frac{1}{i(n_1-n_2)}\{\underbrace{\cos[2\pi(n_1-n_2)]}_{=1} + i\underbrace{\sin[2\pi(n_1-n_2)]}_{=0} - 1\} \\
&= 0
\end{aligned}$$

Part (b)

The hermitian operator in Problem 3.6 is

$$\frac{d^2}{d\phi^2},$$

and its eigenfunctions are $\cos n\phi$ and $\sin n\phi$, where n is a nonnegative integer and $0 \leq \phi \leq 2\pi$.

$$\frac{d^2}{d\phi^2}(\cos n\phi) = -n^2(\cos n\phi)$$

$$\frac{d^2}{d\phi^2}(\sin n\phi) = -n^2(\sin n\phi)$$

The eigenvalue corresponding to $\cos n\phi$ and $\sin n\phi$ is $-n^2$, which is real.

By part (a) of Problem 3.7, any linear combination of $\cos n\phi$ and $\sin n\phi$ is also an eigenfunction of $d^2/d\phi^2$. To show that the eigenfunctions are orthogonal, take the inner product of $A_1 \cos n_1\phi + B_1 \sin n_1\phi$ and $A_2 \cos n_2\phi + B_2 \sin n_2\phi$ (with $n_1 \neq n_2$) and show that it's zero.

$$\begin{aligned}
&\int_0^{2\pi} (A_1 \cos n_1\phi + B_1 \sin n_1\phi)^*(A_2 \cos n_2\phi + B_2 \sin n_2\phi)\, d\phi \\
&= \int_0^{2\pi} (A_1^* \cos n_1\phi + B_1^* \sin n_1\phi)(A_2 \cos n_2\phi + B_2 \sin n_2\phi)\, d\phi \\
&= \int_0^{2\pi} (A_1^* A_2 \cos n_1\phi \cos n_2\phi + A_1^* B_2 \cos n_1\phi \sin n_2\phi + B_1^* A_2 \sin n_1\phi \cos n_2\phi + B_1^* B_2 \sin n_1\phi \sin n_2\phi)\, d\phi \\
&= A_1^* A_2 \int_0^{2\pi} \cos n_1\phi \cos n_2\phi\, d\phi \\
&\quad + A_1^* B_2 \int_0^{2\pi} \cos n_1\phi \sin n_2\phi\, d\phi \\
&\quad + B_1^* A_2 \int_0^{2\pi} \sin n_1\phi \cos n_2\phi\, d\phi \\
&\quad + B_1^* B_2 \int_0^{2\pi} \sin n_1\phi \sin n_2\phi\, d\phi \\
&= A_1^* A_2 \int_0^{2\pi} \frac{1}{2}[\cos(n_1\phi + n_2\phi) + \cos(n_1\phi - n_2\phi)]\, d\phi \\
&\quad + A_1^* B_2 \int_0^{2\pi} \frac{1}{2}[\sin(n_1\phi + n_2\phi) - \sin(n_1\phi - n_2\phi)]\, d\phi \\
&\quad + B_1^* A_2 \int_0^{2\pi} \frac{1}{2}[\sin(n_1\phi + n_2\phi) + \sin(n_1\phi - n_2\phi)]\, d\phi \\
&\quad + B_1^* B_2 \int_0^{2\pi} \frac{1}{2}[\cos(n_1\phi - n_2\phi) - \cos(n_1\phi + n_2\phi)]\, d\phi \\
&= \frac{A_1^* A_2}{2} \left\{ \int_0^{2\pi} \cos[(n_1 + n_2)\phi]\, d\phi + \int_0^{2\pi} \cos[(n_1 - n_2)\phi]\, d\phi \right\} \\
&\quad + \frac{A_1^* B_2}{2} \left\{ \int_0^{2\pi} \sin[(n_1 + n_2)\phi]\, d\phi - \int_0^{2\pi} \sin[(n_1 - n_2)\phi]\, d\phi \right\} \\
&\quad + \frac{B_1^* A_2}{2} \left\{ \int_0^{2\pi} \sin[(n_1 + n_2)\phi]\, d\phi + \int_0^{2\pi} \sin[(n_1 - n_2)\phi]\, d\phi \right\} \\
&\quad + \frac{B_1^* B_2}{2} \left\{ \int_0^{2\pi} \cos[(n_1 - n_2)\phi]\, d\phi - \int_0^{2\pi} \cos[(n_1 + n_2)\phi]\, d\phi \right\} \\
&= \frac{A_1^* A_2}{2}(0) + \frac{A_1^* B_2}{2}(0) + \frac{B_1^* A_2}{2}(0) + \frac{B_1^* B_2}{2}(0) \\
&= 0
\end{aligned}$$