Problem 4.20

Consider the earth–sun system as a gravitational analog to the hydrogen atom.

(a) What is the potential energy function (replacing Equation 4.52)? (Let m_E be the mass of the earth, and M the mass of the sun.)

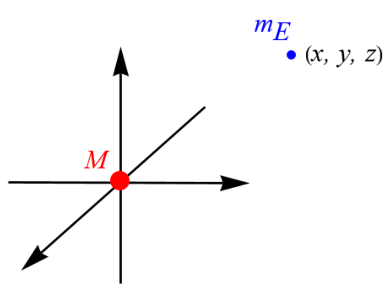
(b) What is the "Bohr radius," a_g, for this system? Work out the actual number.

(c) Write down the gravitational "Bohr formula," and, by equating E_n to the classical energy of a planet in a circular orbit of radius r_o, show that $n = \sqrt{r_o/a_g}$. From this, estimate the quantum number n of the earth.

(d) Suppose the earth made a transition to the next lower level $(n-1)$. How much energy (in Joules) would be released? What would the wavelength of the emitted photon (or, more likely, graviton) be? (Express your answer in light years—is the remarkable answer[28] a coincidence?)

Solution

Consider the earth (with mass $m_E \approx 5.98 \times 10^{24}$ kg) in a circular orbit around the sun (with mass $M \approx 1.99 \times 10^{30}$ kg). Because the sun is roughly 300 000 times more massive than earth, the sun's motion can be neglected to a good approximation. As such, let the sun lie at the origin of space. Ignore the earth's spin.



The earth is somewhere around the sun; solve the Schrödinger equation to determine its wave function.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_E}\nabla^2\Psi + V(x,y,z)\Psi(x,y,z,t)$$

The potential energy function $V(x,y,z)$ is determined from

$$\mathbf{F} = -\nabla V,$$

where $\mathbf{F}$ is given by Newton's law of universal gravitation.

$$-G\frac{(m_E)(M)}{r^2}\,\hat{\mathbf{r}} = -\nabla V,$$

Since the force is only dependent on the spherical coordinate $r = \sqrt{x^2+y^2+z^2}$, the potential energy function is as well.

$$-G\frac{m_E M}{r^2} = -\frac{dV}{dr}$$

[28]Thanks to John Meyer for pointing this out.

Multiply both sides by -1 and then integrate both sides from ∞ to r.

$$\int_\infty^r G\frac{m_E M}{r_0^2}\,dr_0 = \int_\infty^r \frac{dV}{dr}(r_0)\,dr_0$$

Evaluate the integrals.

$$-G\frac{m_E M}{r_0}\bigg|_\infty^r = V(r) - \underbrace{V(\infty)}_{=\,0}$$

As a result,

$$\boxed{V(r) = -G\frac{m_E M}{r},}$$

and Schrödinger's equation becomes

$$\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m_E}\nabla^2\Psi + V(r)\Psi(r,\phi,\theta,t) \\
&= -\frac{\hbar^2}{2m_E}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right] + V(r)\Psi(r,\phi,\theta,t).
\end{aligned}$$

The aim is to solve for $\Psi = \Psi(r,\theta,\phi,t)$ in all of space ($0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$) for $t > 0$. Assuming a product solution of the form $\Psi(r,\theta,\phi,t) = R(r)\Theta(\theta)\xi(\phi)T(t)$ and plugging it into the PDE yields the following system of ODEs (see Problem 4.4).

$$\left.\begin{aligned}
i\hbar\frac{T'(t)}{T(t)} &= E \\
\frac{1}{R(r)}\frac{d}{dr}\left(r^2R'(r)\right) - \frac{2m_E r^2}{\hbar^2}[V(r) - E] &= F \\
\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(\Theta'(\theta)\sin\theta\right) + F\sin^2\theta &= \mathscr{G} \\
-\frac{\xi''(\phi)}{\xi(\phi)} &= \mathscr{G}
\end{aligned}\right\}$$

The normalized products of angular eigenfunctions $\Theta(\theta)\xi(\phi)$ are called the spherical harmonics and are denoted by $Y_\ell^m(\theta,\phi)$. Solutions only exist if $F = \ell(\ell+1)$, where $\ell = 0, 1, 2, \ldots$, and if $\mathscr{G} = m^2$ is an integer.

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,e^{im\phi}P_\ell^m(\cos\theta), \qquad \begin{cases} \ell = 0, 1, 2, \ldots \\ m = -\ell, -\ell+1, \ldots, -1, 0, 1, \ldots, \ell-1, \ell \end{cases}$$

With these results the equation for $R(r)$ becomes

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2R'(r)\right) - \frac{2m_E r^2}{\hbar^2}\left(-\frac{Gm_E M}{r} - E\right) = \ell(\ell+1)$$

$$\frac{d}{dr}\left[r^2\frac{dR}{dr}(r)\right] + \left[2\left(\frac{Gm_E^2 M}{\hbar^2}\right)r + \frac{2m_E r^2}{\hbar^2}E\right]R(r) - \ell(\ell+1)R(r) = 0. \tag{1}$$

Note that since we're interested in the bound states (the sun and earth paired together), $E < 0$. Also, the grouping of constants in parentheses is $1/a_g$, where a_g is the gravitational Bohr radius.

$$\boxed{a_g = \frac{\hbar^2}{Gm_E^2 M} \approx \frac{(1.054571726 \times 10^{-34}\ \mathrm{J \cdot s})^2}{\left(6.673 \times 10^{-11}\ \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}\right)(5.98 \times 10^{24}\ \mathrm{kg})^2 (1.99 \times 10^{30}\ \mathrm{kg})} \approx 2.34 \times 10^{-138}\ \mathrm{m}}$$

Make the change of variables,

$$s = \kappa r, \quad \text{where } \kappa = \frac{\sqrt{-8m_E E}}{\hbar}.$$

Consequently, equation (1) turns into

$$\frac{ds}{dr}\frac{d}{ds}\left[\left(\frac{s}{\kappa}\right)^2 \frac{ds}{dr}\frac{d}{ds} R\left(\frac{s}{\kappa}\right)\right] + \left[\frac{2}{a_g}\left(\frac{s}{\kappa}\right) + \frac{2m_E s^2}{\hbar^2 \kappa^2} E\right] R\left(\frac{s}{\kappa}\right) - \ell(\ell+1) R\left(\frac{s}{\kappa}\right) = 0.$$

Use a new dependent variable,

$$w(s) = R\left(\frac{s}{\kappa}\right),$$

and simplify the left side.

$$\kappa \frac{d}{ds}\left[\left(\frac{s^2}{\kappa^2}\right)\kappa \frac{dw}{ds}\right] + \left[\frac{2}{a_g}\left(\frac{s}{\kappa}\right) + \frac{2m_E s^2}{\hbar^2}\left(-\frac{\hbar^2}{8m_E}\right)\right] w(s) - \ell(\ell+1) w(s) = 0$$

$$\frac{d}{ds}\left(s^2 \frac{dw}{ds}\right) + \left[\frac{2s}{a_g \kappa} - \frac{s^2}{4} - \ell(\ell+1)\right] w(s) = 0$$

Make another change of variables.

$$w(s) = s^\ell e^{-s/2} u(s)$$

As a result,

$$\begin{aligned}
0 &= \frac{d}{ds}\left\{s^2 \frac{d}{ds}[s^\ell e^{-s/2} u(s)]\right\} + \left[\frac{2s}{a_g\kappa} - \frac{s^2}{4} - \ell(\ell+1)\right] s^\ell e^{-s/2} u(s) \\
&= \frac{d}{ds}\left\{s^2 \left[\ell s^{\ell-1} e^{-s/2} u(s) + s^\ell\left(-\frac{1}{2}\right) e^{-s/2} u(s) + s^\ell e^{-s/2}\frac{du}{ds}\right]\right\} + \left[\frac{2s}{a_g\kappa} - \frac{s^2}{4} - \ell(\ell+1)\right] s^\ell e^{-s/2} u(s) \\
&= \frac{d}{ds}\left(\ell s^{\ell+1} e^{-s/2} u(s) - \frac{1}{2} s^{\ell+2} e^{-s/2} u(s) + s^{\ell+2} e^{-s/2}\frac{du}{ds}\right) + \left[\frac{2s}{a_g\kappa} - \frac{s^2}{4} - \ell(\ell+1)\right] s^\ell e^{-s/2} u(s) \\
&= \left[\cancel{\ell(\ell+1) s^\ell e^{-s/2} u(s)} + \ell s^{\ell+1}\left(-\frac{1}{2}\right) e^{-s/2} u(s) + \ell s^{\ell+1} e^{-s/2}\frac{du}{ds}\right] \\
&\qquad - \frac{1}{2}\left[(\ell+2) s^{\ell+1} e^{-s/2} u(s) + \cancel{s^{\ell+2}\left(-\frac{1}{2}\right) e^{-s/2} u(s)} + s^{\ell+2} e^{-s/2}\frac{du}{ds}\right] \\
&\qquad\quad + \left[(\ell+2) s^{\ell+1} e^{-s/2}\frac{du}{ds} + s^{\ell+2}\left(-\frac{1}{2}\right) e^{-s/2}\frac{du}{ds} + s^{\ell+2} e^{-s/2}\frac{d^2u}{ds^2}\right] \\
&\qquad\qquad + \frac{2}{a_g\kappa} s^{\ell+1} e^{-s/2} u(s) - \cancel{\frac{1}{4} s^{\ell+2} e^{-s/2} u(s)} - \cancel{\ell(\ell+1) s^\ell e^{-s/2} u(s)} \\
&= s^{\ell+2} e^{-s/2}\frac{d^2u}{ds^2} + (2\ell + 2 - s) s^{\ell+1} e^{-s/2}\frac{du}{ds} + \left(\frac{2}{a_g\kappa} - \ell - 1\right) s^{\ell+1} e^{-s/2} u(s).
\end{aligned}$$

Multiply both sides by $e^{s/2}$.

$$s^{\ell+2}\frac{d^2u}{ds^2} + (2\ell + 2 - s)s^{\ell+1}\frac{du}{ds} + \left(\frac{2}{a_g\kappa} - \ell - 1\right)s^{\ell+1}u(s) = 0$$

Divide both sides by $s^{\ell+1}$.

$$s\frac{d^2u}{ds^2} + [(2\ell+1) + 1 - s]\frac{du}{ds} + \left(\frac{2}{a_g\kappa} - \ell - 1\right)u(s) = 0, \quad 0 < s < \infty$$

This is the generalized Laguerre differential equation. Normalizable solutions exist only if the quantity in parentheses multiplying $u(s)$ is a nonnegative integer $(0, 1, 2, \ldots)$. It's this fact that allows us to determine the eigenenergies of the bound states of the earth-sun system. Let N be the nonnegative integer.

$$\frac{2}{a_g\kappa} - \ell - 1 = N \quad \rightarrow \quad \frac{2}{a_g\kappa} = N + \ell + 1$$

The number on the right side is a positive integer $(1, 2, \ldots)$ and is denoted by n.

$$\frac{2}{a_g\kappa} = n \quad \rightarrow \quad 2\left(\frac{Gm_E^2M}{\hbar^2}\right)\left(\frac{\hbar}{\sqrt{-8m_EE}}\right) = n$$

Solve for E.

$$E_n = -\frac{G^2M^2m_E^3}{2\hbar^2n^2} = -\left[\frac{m_E}{2\hbar^2}(Gm_EM)^2\right]\frac{1}{n^2}, \quad n = 1, 2, \ldots$$

The classical energy of earth is the sum of its kinetic and potential energies. Let r_o be the distance from the center of the earth to the center of the sun.

$$\begin{aligned} E_c &= \text{KE} + \text{PE} \\ &= \frac{1}{2}m_Ev^2 + \left(-\frac{Gm_EM}{r_o}\right) \end{aligned} \tag{2}$$

In order to simplify this formula, apply Newton's second law to the earth in the radial direction. The only force acting on the earth is the gravitational force from the sun.

$$\sum \mathbf{F} = m_E\mathbf{a} \quad \Rightarrow \quad -\frac{Gm_EM}{r_o^2} = m_E\left(-\frac{v^2}{r_o}\right) \quad \rightarrow \quad \frac{Gm_EM}{2r_o} = \frac{1}{2}m_Ev^2$$

Consequently, equation (2) becomes

$$\begin{aligned} E_c &= \frac{Gm_EM}{2r_o} + \left(-\frac{Gm_EM}{r_o}\right) \\ &= -\frac{Gm_EM}{2r_o}. \end{aligned}$$

Now set E_c equal to E_n, solve for n^2, and write it in terms of the gravitational Bohr radius.

$$E_c = E_n$$

$$-\frac{Gm_E M}{2r_o} = -\frac{G^2M^2m_E^3}{2\hbar^2 n^2}$$

$$n^2 = \frac{Gm_E^2 M}{\hbar^2} r_o = \left(\frac{1}{a_g}\right) r_o \quad \rightarrow \quad \boxed{n = \sqrt{\frac{r_o}{a_g}}}$$

$$n = \sqrt{\frac{Gm_E^2 M}{\hbar^2} r_o}$$

Therefore, the earth's quantum number is

$$\boxed{n \approx \sqrt{\frac{\left(6.673 \times 10^{-11} \ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)(5.98 \times 10^{24} \text{ kg})^2 (1.99 \times 10^{30} \text{ kg})}{(1.054571726 \times 10^{-34} \text{ J} \cdot \text{s})^2}(1.496 \times 10^{11} \text{ m})} \approx 2.53 \times 10^{74}.}$$

If the earth transitions from an initial stationary state with $n = n_i$ to a final stationary state with $n = n_f$ in which $n_i > n_f$, then a graviton will be emitted with energy ΔE.

$$\begin{aligned}
\Delta E &= E_{n_i} - E_{n_f} \\
&= -\frac{G^2M^2m_E^3}{2\hbar^2 n_i^2} + \frac{G^2M^2m_E^3}{2\hbar^2 n_f^2} \\
&= \frac{G^2M^2m_E^3}{2\hbar^2}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)
\end{aligned}$$

For the particular case that $n_i = n$ and $n_f = n - 1$,

$$\Delta E = \frac{G^2M^2m_E^3}{2\hbar^2}\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right].$$

Because of how large n is, it's necessary to rewrite the expression in square brackets.

$$\begin{aligned}
\Delta E &= \frac{G^2M^2m_E^3}{2\hbar^2}\left[\frac{1}{n^2\left(1-\frac{1}{n}\right)^2} - \frac{1}{n^2}\right] \\
&= \frac{G^2M^2m_E^3}{2\hbar^2}\frac{1}{n^2}\left[\frac{1}{\left(1-\frac{1}{n}\right)^2} - 1\right] \\
&= \frac{G^2M^2m_E^3}{2\hbar^2 n^2}\left[\left(1-\frac{1}{n}\right)^{-2} - 1\right] \\
&= \frac{G^2M^2m_E^3}{2\hbar^2 n^2}\left[1 + (-2)\left(-\frac{1}{n}\right) + \frac{(-2)(-2-1)}{2!}\left(-\frac{1}{n}\right)^2 + \frac{(-2)(-2-1)(-2-2)}{3!}\left(-\frac{1}{n}\right)^3 + \cdots - 1\right] \\
&= \frac{G^2M^2m_E^3}{2\hbar^2 n^2}\left[2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right)^2 + 4\left(\frac{1}{n}\right)^3 + 5\left(\frac{1}{n}\right)^4 + \cdots\right]
\end{aligned}$$

The higher-order terms are extremely negligible compared to the first in the square brackets.

$$\begin{aligned}
\Delta E &= \frac{G^2M^2m_E^3}{2\hbar^2n^2}\left[2\left(\frac{1}{n}\right)\right] \\
&= \frac{G^2M^2m_E^3}{\hbar^2}\left(\frac{1}{n^2}\right)\left(\frac{1}{n}\right) \\
&= \frac{G^2M^2m_E^3}{\hbar^2}\left(\frac{\hbar^2}{Gm_E^2Mr_o}\right)\left(\frac{\hbar}{m_E}\frac{1}{\sqrt{GMr_o}}\right) \\
&= \hbar\sqrt{\frac{GM}{r_o^3}} \qquad (3) \\
&\approx (1.054571726\times 10^{-34}\text{ J}\cdot\text{s})\sqrt{\frac{\left(6.673\times 10^{-11}\ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)(1.99\times 10^{30}\text{ kg})}{(1.496\times 10^{11}\text{ m})^3}}
\end{aligned}$$

Therefore,

$$\boxed{\Delta E \approx 2.10\times 10^{-41}\text{ J.}}$$

Now calculate the wavelength from equation (3).

$$\Delta E = \hbar\sqrt{\frac{GM}{r_o^3}}$$

$$h\nu = \frac{h}{2\pi}\sqrt{\frac{GM}{r_o^3}} \quad\rightarrow\quad \nu = \frac{1}{2\pi}\sqrt{\frac{GM}{r_o^3}}$$

$$h\left(\frac{c}{\lambda}\right) = \frac{h}{2\pi}\sqrt{\frac{GM}{r_o^3}}$$

Therefore, solving for λ,

$$\boxed{\lambda = 2\pi c\sqrt{\frac{r_o^3}{GM}} \approx 2\pi\left(299\,792\,458\ \frac{\text{m}}{\text{s}}\right)\sqrt{\frac{(1.496\times 10^{11}\text{ m})^3}{\left(6.673\times 10^{-11}\ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)(1.99\times 10^{30}\text{ kg})}} \approx 9.46\times 10^{15}\text{ m.}}$$

Note that one light year is the distance light travels in one year.

$$\begin{aligned}
1\text{ light year} &= c(1\text{ year}) \\
&\approx \left(299\,792\,458\ \frac{\text{m}}{\text{s}}\right)\left(1\text{ year}\times\frac{365.25\text{ days}}{1\text{ year}}\times\frac{24\text{ hr}}{1\text{ day}}\times\frac{60\text{ min}}{1\text{ hr}}\times\frac{60\text{ s}}{1\text{ min}}\right) \\
&\approx 9.46\times 10^{15}\text{ m}
\end{aligned}$$

That means

$$\boxed{\lambda \approx 9.46\times 10^{15}\text{ m}\times\frac{1\text{ light year}}{9.46\times 10^{15}\text{ m}} \approx 1.00\text{ light year.}}$$

This remarkable answer is not a coincidence; the wavelength of the graviton is how far light travels in the time it takes the earth to make one revolution around the sun.