Problem 4.25

(a) What is $L_+Y_\ell^\ell$? (No calculation allowed!)

(b) Use the result of (a), together with Equation 4.130 and the fact that $L_zY_\ell^\ell = \hbar\ell Y_\ell^\ell$, to determine $Y_\ell^\ell(\theta,\phi)$, up to a normalization constant.

(c) Determine the normalization constant by direct integration. Compare your final answer to what you got in Problem 4.7.

Solution

Part (a)

According to the result of Problem 4.21, applying the raising operator to a function gives

$$\begin{aligned} L_+f_\ell^m &= A_\ell^m f_\ell^{m+1} \\ &= \hbar\sqrt{\ell(\ell+1)-m(m+1)}\, f_\ell^{m+1}. \end{aligned}$$

Apply the raising operator to $Y_\ell^\ell(\theta,\phi)$.

$$\begin{aligned} L_+Y_\ell^\ell &= \hbar\sqrt{\ell(\ell+1)-\ell(\ell+1)}\, Y_\ell^{\ell+1} \\ &= \hbar(0)Y_\ell^{\ell+1} \\ &= 0 \end{aligned}$$

This makes sense, as $m=\ell$ is the highest that m can go for a given ℓ.

Part (b)

The goal in this part is to solve two simultaneous equations for $Y_\ell^\ell(\theta,\phi)$.

$$\begin{cases} L_+Y_\ell^\ell = 0 \\ L_zY_\ell^\ell = \hbar\ell Y_\ell^\ell \end{cases}$$

$$\begin{cases} \hbar e^{i\phi}\left(\dfrac{\partial}{\partial\theta} + i\cot\theta\dfrac{\partial}{\partial\phi}\right)Y_\ell^\ell = 0 \\ \left(-i\hbar\dfrac{\partial}{\partial\phi}\right)Y_\ell^\ell = \hbar\ell Y_\ell^\ell \end{cases}$$

$$\begin{cases} \dfrac{\partial Y_\ell^\ell}{\partial\theta} + i\cot\theta\dfrac{\partial Y_\ell^\ell}{\partial\phi} = 0 \\ \dfrac{\partial Y_\ell^\ell}{\partial\phi} = i\ell Y_\ell^\ell \end{cases}$$

For the sake of convenience, solve the second PDE first.

$$\frac{\partial Y_\ell^\ell}{\partial \phi} - i\ell Y_\ell^\ell = 0$$

Multiply both sides by the integrating factor,

$$\exp\left(\int -i\ell\, d\phi\right) = e^{-i\ell\phi},$$

to make the left side a partial derivative by the product rule.

$$e^{-i\ell\phi}\frac{\partial Y_\ell^\ell}{\partial \phi} - i\ell e^{-i\ell\phi} Y_\ell^\ell = 0$$

$$\frac{\partial}{\partial \phi}(e^{-i\ell\phi} Y_\ell^\ell) = 0$$

Integrate both sides partially with respect to ϕ.

$$e^{-i\ell\phi} Y_\ell^\ell = f(\theta)$$

Here $f(\theta)$ is an arbitrary function. Multiply both sides by $e^{i\ell\phi}$.

$$Y_\ell^\ell(\theta, \phi) = f(\theta) e^{i\ell\phi}$$

Substitute this result back into the first equation to determine $f(\theta)$.

$$\frac{\partial}{\partial \theta}[f(\theta)e^{i\ell\phi}] + i\cot\theta \frac{\partial}{\partial \phi}[f(\theta)e^{i\ell\phi}] = 0$$

$$e^{i\ell\phi}\frac{df}{d\theta} + i\cot\theta[f(\theta)i\ell e^{i\ell\phi}] = 0$$

$$\frac{df}{d\theta} - (\ell\cot\theta)f = 0$$

Multiply both sides by the integrating factor,

$$\exp\left(\int -\ell\cot\theta\, d\theta\right) = e^{-\ell\ln\sin\theta} = e^{\ln(\sin\theta)^{-\ell}} = (\sin\theta)^{-\ell},$$

to make the left side a derivative by the product rule.

$$(\sin\theta)^{-\ell}\frac{df}{d\theta} - (\ell\cot\theta)(\sin\theta)^{-\ell} f = 0$$

$$\frac{d}{d\theta}[(\sin\theta)^{-\ell} f] = 0$$

Integrate both sides with respect to θ.

$$(\sin\theta)^{-\ell} f = A$$

Multiply both sides by $\sin^\ell\theta$.

$$f(\theta) = A\sin^\ell\theta$$

Therefore,

$$Y_\ell^\ell(\theta, \phi) = Ae^{i\ell\phi}\sin^\ell\theta.$$

Part (c)

The normalization of the stationary states requires that

$$\begin{aligned}
1 = \iiint\limits_{\text{all space}} |\Psi(r,\theta,\phi,t)|^2\,d\mathcal{V} &= \iiint\limits_{\text{all space}} |R(r)\Theta(\theta)\xi(\phi)T(t)|^2\,d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} \left|R(r)Y_\ell^m(\theta,\phi)e^{-iEt/\hbar}\right|^2 d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} |R(r)|^2\,|Y_\ell^m(\theta,\phi)|^2\,d\mathcal{V} \\
&= \int_0^\pi \int_0^{2\pi} \int_0^\infty |R(r)|^2\,|Y_\ell^m(\theta,\phi)|^2\,(r^2\sin\theta\,dr\,d\phi\,d\theta) \\
&= \underbrace{\left[\int_0^\infty r^2|R(r)|^2\,dr\right]}_{=1}\underbrace{\left[\int_0^\pi \int_0^{2\pi} |Y_\ell^m(\theta,\phi)|^2\sin\theta\,d\phi\,d\theta\right]}_{=1}.
\end{aligned}$$

Determine the constant A by requiring $Y_\ell^\ell(\theta,\phi)$ to be normalized.

$$\begin{aligned}
1 &= \int_0^\pi \int_0^{2\pi} |Y_\ell^\ell(\theta,\phi)|^2\sin\theta\,d\phi\,d\theta \\
&= \int_0^\pi \int_0^{2\pi} |Ae^{i\ell\phi}\sin^\ell\theta|^2\sin\theta\,d\phi\,d\theta \\
&= \int_0^\pi \int_0^{2\pi} |A|^2\sin^{2\ell}\theta\sin\theta\,d\phi\,d\theta \\
&= |A|^2\left(\int_0^{2\pi} d\phi\right)\int_0^\pi \sin^{2\ell}\theta\sin\theta\,d\theta \\
&= 2\pi|A|^2\int_0^\pi (\sin^2\theta)^\ell\sin\theta\,d\theta \\
&= 2\pi|A|^2\int_0^\pi (1-\cos^2\theta)^\ell\sin\theta\,d\theta
\end{aligned}$$

Make the following substitution.

$$u = \cos\theta$$

$$du = -\sin\theta\,d\theta \quad \rightarrow \quad -du = \sin\theta\,d\theta$$

As a result,

$$1 = 2\pi|A|^2 \int_{\cos 0}^{\cos \pi} (1-u^2)^\ell (-du)$$

$$= 2\pi|A|^2 \int_{-1}^{1} (1-u^2)^\ell \, du$$

$$= 4\pi|A|^2 \int_{0}^{1} (1-u^2)^\ell \, du.$$

Use the binomial theorem to expand the integrand. Since ℓ is an integer, the series is finite.

$$1 = 4\pi|A|^2 \int_0^1 \sum_{k=0}^{\ell} \frac{\ell!}{k!(\ell-k)!}(-u^2)^k \, du$$

$$= 4\pi|A|^2 \ell! \int_0^1 \sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!} u^{2k} \, du$$

$$= 4\pi|A|^2 \ell! \sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!} \left. \frac{u^{2k+1}}{2k+1} \right|_0^1$$

$$= 4\pi|A|^2 \ell! \sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!(2k+1)}$$

In order to find the sum, evaluate it for several values of ℓ until a pattern becomes apparent.

$$\ell = 0: \quad \sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!(2k+1)} = 1$$

$$\ell = 1: \quad \sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!(2k+1)} = \frac{2}{3}$$

$$\ell = 2: \quad \sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!(2k+1)} = \frac{4}{15}$$

$$\ell = 3: \quad \sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!(2k+1)} = \frac{8}{105}$$

$$\ell = 4: \quad \sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!(2k+1)} = \frac{16}{945}$$

$$\ell = 5: \quad \sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!(2k+1)} = \frac{32}{10\,395}$$

Generally, it is

$$\sum_{k=0}^{\ell} \frac{(-1)^k}{k!(\ell-k)!(2k+1)} = \frac{2^\ell}{(2\ell+1)!!}$$

$$= \frac{2^\ell}{(2\ell+1)(2\ell-1)(2\ell-3)\cdots 5\cdot 3\cdot 1}$$

$$= \frac{2^\ell \cdot (2\ell)(2\ell-2)(2\ell-4)\cdots 4\cdot 2}{(2\ell+1)(2\ell)(2\ell-1)(2\ell-2)(2\ell-3)\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{2^\ell \cdot 2^\ell(\ell)(\ell-1)(\ell-2)\cdots 2\cdot 1}{(2\ell+1)(2\ell)(2\ell-1)(2\ell-2)(2\ell-3)\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{2^{2\ell}\ell!}{(2\ell+1)!},$$

which means

$$1 = 4\pi|A|^2\ell!\left[\frac{2^{2\ell}\ell!}{(2\ell+1)!}\right]$$

$$= 4\pi|A|^2\frac{2^{2\ell}(\ell!)^2}{(2\ell+1)!}$$

$$= 4\pi|A|^2\frac{(2^\ell\ell!)^2}{(2\ell+1)!}.$$

Solve for $|A|^2$.

$$|A|^2 = \frac{1}{(2^\ell\ell!)^2}\frac{(2\ell+1)!}{4\pi}$$

Take the square root of both sides.

$$|A| = \frac{1}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}$$

Remove the modulus by placing an arbitrary phase factor on the right side.

$$\boxed{A = \frac{e^{i\beta}}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}}$$

In order to make this result equivalent to the one from Problem 4.7, set $\beta = \pi\ell$.

$$A = \frac{e^{i\pi\ell}}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}} = \frac{(e^{i\pi})^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}} = \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}$$

Therefore,

$$Y_\ell^\ell(\theta,\phi) = \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\, e^{i\ell\phi}\sin^\ell\theta.$$