Problem 4.31

For the most general normalized spinor χ (Equation 4.139), compute $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$, $\langle S_x^2 \rangle$, $\langle S_y^2 \rangle$, and $\langle S_z^2 \rangle$. Check that $\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \langle S^2 \rangle$.

Solution

The general normalized state of a particle with spin 1/2 is

$$\chi = \begin{bmatrix} a \\ b \end{bmatrix}$$

and is subject to the condition that $|a|^2 + |b|^2 = 1$. Calculate the expectation value of S_x.

$$\begin{aligned}
\langle S_x \rangle = \frac{\langle \chi \,|\, S_x \,|\, \chi \rangle}{\langle \chi \,|\, \chi \rangle} = \frac{\chi^\dagger S_x \chi}{1} &= \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} \\
&= \frac{\hbar}{2} (a^* b + b^* a) \\
&= \hbar \left[\frac{a^* b + (a^* b)^*}{2} \right] \\
&= \hbar \, \mathrm{Re}\,(a^* b)
\end{aligned}$$

Calculate the expectation value of S_x^2.

$$\begin{aligned}
\langle S_x^2 \rangle = \frac{\langle \chi \,|\, S_x^2 \,|\, \chi \rangle}{\langle \chi \,|\, \chi \rangle} = \frac{\chi^\dagger S_x^2 \chi}{1} &= \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} \\
&= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \frac{\hbar^2}{4} (a^* a + b^* b) \\
&= \frac{\hbar^2}{4}
\end{aligned}$$

Calculate the expectation value of S_y.

$$\langle S_y \rangle = \frac{\langle \chi \,|\, S_y \,|\, \chi \rangle}{\langle \chi \,|\, \chi \rangle} = \frac{\chi^\dagger S_y \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} -ib \\ ia \end{bmatrix}$$

$$= \frac{\hbar}{2}(-ia^*b + ib^*a)$$

$$= \hbar \left[\frac{a^*b - (a^*b)^*}{2i} \right]$$

$$= \hbar \, \mathrm{Im}\,(a^*b)$$

Calculate the expectation value of S_y^2.

$$\langle S_y^2 \rangle = \frac{\langle \chi \,|\, S_y^2 \,|\, \chi \rangle}{\langle \chi \,|\, \chi \rangle} = \frac{\chi^\dagger S_y^2 \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} -ib \\ ia \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar^2}{4}(a^*a + b^*b)$$

$$= \frac{\hbar^2}{4}(|a|^2 + |b|^2)$$

$$= \frac{\hbar^2}{4}$$

Calculate the expectation value of S_z.

$$\langle S_z \rangle = \frac{\langle \chi \,|\, S_z \,|\, \chi \rangle}{\langle \chi \,|\, \chi \rangle} = \frac{\chi^\dagger S_z \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} a \\ -b \end{bmatrix}$$

$$= \frac{\hbar}{2}(a^* a - b^* b)$$

$$= \frac{\hbar}{2}(|a|^2 - |b|^2)$$

Calculate the expectation value of S_z^2.

$$\langle S_z^2 \rangle = \frac{\langle \chi \,|\, S_z^2 \,|\, \chi \rangle}{\langle \chi \,|\, \chi \rangle} = \frac{\chi^\dagger S_z^2 \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ -b \end{bmatrix}$$

$$= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{\hbar^2}{4}(a^* a + b^* b)$$

$$= \frac{\hbar^2}{4}(|a|^2 + |b|^2)$$

$$= \frac{\hbar^2}{4}$$

The reason $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \hbar^2/4$ is because each Pauli matrix is its own inverse.

$$\langle S_j^2 \rangle = \langle \chi \,|\, S_j^2 \,|\, \chi \rangle = \left\langle \chi \,\middle|\, \frac{\hbar}{2}\sigma_j \frac{\hbar}{2}\sigma_j \,\middle|\, \chi \right\rangle = \frac{\hbar^2}{4} \langle \chi \,|\, \sigma_j \sigma_j^{-1} \,|\, \chi \rangle = \frac{\hbar^2}{4} \langle \chi \,|\, \mathsf{I} \,|\, \chi \rangle = \frac{\hbar^2}{4} \langle \chi \,|\, \chi \rangle$$

$$= \frac{\hbar^2}{4}(1)$$

$$= \frac{\hbar^2}{4}$$

The defining eigenvalue problem for S^2 is in Equation 4.135 on page 166.

$$S^2|s\,m_s\rangle = \hbar^2 s(s+1)|s\,m_s\rangle$$

Premultiply both sides by $\langle s\,m_s|$.

$$\langle s\,m_s\,|\,S^2\,|s\,m_s\rangle = \langle s\,m_s\,|\,\hbar^2 s(s+1)\,|s\,m_s\rangle$$

$$\langle s\,m_s\,|\,S^2\,|s\,m_s\rangle = \hbar^2 s(s+1)\langle s\,m_s\,|\,s\,m_s\rangle$$

$$\frac{\langle s\,m_s\,|\,S^2\,|s\,m_s\rangle}{\langle s\,m_s\,|\,s\,m_s\rangle} = \hbar^2 s(s+1)$$

$$\langle S^2\rangle = \hbar^2 s(s+1)$$

For spin 1/2,

$$\langle S^2\rangle = \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3\hbar^2}{4},$$

so

$$\langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle = \frac{\hbar^2}{4} + \frac{\hbar^2}{4} + \frac{\hbar^2}{4} = \frac{3\hbar^2}{4} = \langle S^2\rangle.$$