Problem 4.33

Construct the matrix S_r representing the component of spin angular momentum along an arbitrary direction $\hat{r}$. Use spherical coordinates, for which

$$\hat{r} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}. \tag{4.154}$$

Find the eigenvalues and (normalized) eigenspinors of S_r. *Answer:*

$$\chi_+^{(r)} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}; \quad \chi_-^{(r)} = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \tag{4.155}$$

Note: You're always free to multiply by an arbitrary phase factor—say, $e^{i\phi}$—so your answer may not *look* exactly the same as mine.

Solution

Start with the general formula for the matrix of spin angular momentum in terms of the Pauli matrices.

$$\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$$

Take the dot product of both sides with $\hat{\mathbf{r}}$, the unit vector pointing away from the origin.

$$\mathbf{S}\cdot\hat{\mathbf{r}} = \frac{\hbar}{2}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}$$

$$\begin{aligned}
\mathsf{S}_r &= \frac{\hbar}{2}\langle\sigma_x, \sigma_y, \sigma_z\rangle \cdot \langle\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\rangle \\
&= \frac{\hbar}{2}(\sigma_x\sin\theta\cos\phi + \sigma_y\sin\theta\sin\phi + \sigma_z\cos\theta) \\
&= \frac{\hbar}{2}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\sin\theta\cos\phi + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\sin\theta\sin\phi + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\cos\theta\right) \\
&= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{bmatrix} \\
&= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & (\cos\phi - i\sin\phi)\sin\theta \\ (\cos\phi + i\sin\phi)\sin\theta & -\cos\theta \end{bmatrix} \\
&= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix}
\end{aligned}$$

Therefore,

$$\mathsf{S}_r = \begin{bmatrix} \frac{\hbar}{2}\cos\theta & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & -\frac{\hbar}{2}\cos\theta \end{bmatrix}.$$

Determine the eigenvalues of S_r.

$$\det(\mathsf{S}_r - \lambda\mathsf{I}) = 0$$

$$\begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} = 0$$

$$\left(\frac{\hbar}{2}\cos\theta - \lambda\right)\left(-\frac{\hbar}{2}\cos\theta - \lambda\right) - \left(\frac{\hbar}{2}e^{-i\phi}\sin\theta\right)\left(\frac{\hbar}{2}e^{i\phi}\sin\theta\right) = 0$$

$$\left(\lambda - \frac{\hbar}{2}\cos\theta\right)\left(\lambda + \frac{\hbar}{2}\cos\theta\right) - \frac{\hbar^2}{4}\sin^2\theta = 0$$

$$\lambda^2 - \frac{\hbar^2}{4}\cos^2\theta - \frac{\hbar^2}{4}\sin^2\theta = 0$$

$$\lambda^2 = \frac{\hbar^2}{4}$$

$$\lambda = \pm\frac{\hbar}{2}$$

Let

$$\lambda_- = -\frac{\hbar}{2} \quad \text{and} \quad \lambda_+ = +\frac{\hbar}{2}.$$

Determine the eigenvectors (or rather eigenspinors in this context) associated with these eigenvalues.

$$(\mathsf{S}_r - \lambda_-\mathsf{I})\chi_-^{(r)} = \mathsf{0}$$

$$\begin{bmatrix} \frac{\hbar}{2}\cos\theta + \frac{\hbar}{2} & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & -\frac{\hbar}{2}\cos\theta + \frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \left(\frac{\hbar}{2}\cos\theta + \frac{\hbar}{2}\right)x_1 + \frac{\hbar}{2}e^{-i\phi}\sin\theta\, x_2 = 0 \\ \frac{\hbar}{2}e^{i\phi}\sin\theta\, x_1 + \left(-\frac{\hbar}{2}\cos\theta + \frac{\hbar}{2}\right)x_2 = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} (1 + \cos\theta)x_1 + e^{-i\phi}\sin\theta\, x_2 = 0 \\ e^{i\phi}\sin\theta\, x_1 + (1 - \cos\theta)x_2 = 0 \end{aligned}\right\}$$

$$x_1 = -\frac{e^{-i\phi}\sin\theta}{1 + \cos\theta}x_2 = -e^{-i\phi}\tan\frac{\theta}{2}\, x_2$$

$$(\mathsf{S}_r - \lambda_+\mathsf{I})\chi_+^{(r)} = \mathsf{0}$$

$$\begin{bmatrix} \frac{\hbar}{2}\cos\theta - \frac{\hbar}{2} & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & -\frac{\hbar}{2}\cos\theta - \frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \left(\frac{\hbar}{2}\cos\theta - \frac{\hbar}{2}\right)x_1 + \frac{\hbar}{2}e^{-i\phi}\sin\theta\, x_2 = 0 \\ \frac{\hbar}{2}e^{i\phi}\sin\theta\, x_1 + \left(-\frac{\hbar}{2}\cos\theta - \frac{\hbar}{2}\right)x_2 = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} (\cos\theta - 1)x_1 + e^{-i\phi}\sin\theta\, x_2 = 0 \\ e^{i\phi}\sin\theta\, x_1 - (1 + \cos\theta)x_2 = 0 \end{aligned}\right\}$$

$$x_2 = \frac{e^{i\phi}\sin\theta}{1 + \cos\theta}x_1 = e^{i\phi}\tan\frac{\theta}{2}\, x_1$$

Consequently,

$$\chi_-^{(r)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \, x_2 \\ \\ x_2 \end{bmatrix} \qquad \chi_+^{(r)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \\ e^{i\phi} \tan \frac{\theta}{2} \, x_1 \end{bmatrix}.$$

On the left, set $x_2 = -A_2 \cos \frac{\theta}{2}$; on the right, set $x_1 = A_1 \cos \frac{\theta}{2}$.

$$\chi_-^{(r)} = \begin{bmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \left(-A_2 \cos \frac{\theta}{2}\right) \\ \\ -A_2 \cos \frac{\theta}{2} \end{bmatrix} \qquad \chi_+^{(r)} = \begin{bmatrix} A_1 \cos \frac{\theta}{2} \\ \\ e^{i\phi} \tan \frac{\theta}{2} \left(A_1 \cos \frac{\theta}{2}\right) \end{bmatrix}$$

$$\chi_-^{(r)} = \begin{bmatrix} A_2 e^{-i\phi} \sin \frac{\theta}{2} \\ \\ -A_2 \cos \frac{\theta}{2} \end{bmatrix} \qquad \chi_+^{(r)} = \begin{bmatrix} A_1 \cos \frac{\theta}{2} \\ \\ A_1 e^{i\phi} \sin \frac{\theta}{2} \end{bmatrix}$$

Determine A_2 and A_1 by requiring the eigenvectors to be normalized.

$$\left| A_2 e^{-i\phi} \sin \frac{\theta}{2} \right|^2 + \left| -A_2 \cos \frac{\theta}{2} \right|^2 = 1 \qquad \left| A_1 \cos \frac{\theta}{2} \right|^2 + \left| A_1 e^{i\phi} \sin \frac{\theta}{2} \right|^2 = 1$$

$$A_2^2 \sin^2 \frac{\theta}{2} + A_2^2 \cos^2 \frac{\theta}{2} = 1 \qquad A_1^2 \cos^2 \frac{\theta}{2} + A_1^2 \sin^2 \frac{\theta}{2} = 1$$

$$A_2^2 = 1 \qquad A_1^2 = 1$$

$$A_2 = 1 \qquad A_1 = 1$$

Therefore, the eigenvalues and the corresponding normalized eigenspinors of S_r are

$$\begin{cases} \lambda_- = -\dfrac{\hbar}{2} \\ \\ \chi_-^{(r)} = \begin{bmatrix} e^{-i\phi} \sin \frac{\theta}{2} \\ \\ -\cos \frac{\theta}{2} \end{bmatrix} \end{cases} \quad \text{and} \quad \begin{cases} \lambda_+ = +\dfrac{\hbar}{2} \\ \\ \chi_+^{(r)} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \\ e^{i\phi} \sin \frac{\theta}{2} \end{bmatrix} \end{cases}.$$