Problem 4.35

In Example 4.3:

(a) If you measured the component of spin angular momentum along the x direction, at time t, what is the probability that you would get $+\hbar/2$?

(b) Same question, but for the y component.

(c) Same, for the z component.

Solution

In Example 4.3 there's a charged particle with spin 1/2 at rest that's placed in a uniform magnetic field: $\mathbf{B} = B_0\hat{\mathbf{z}}$. The Hamiltonian matrix is given in Equation 4.158 on page 172.

$$\begin{aligned} \mathsf{H} &= -\gamma\,\mathbf{B}\cdot\mathsf{S} \qquad (4.158) \\ &= -\gamma(B_0\hat{\mathbf{z}})\cdot\mathsf{S} \\ &= -\gamma B_0 \mathsf{S}_z \\ &= -\gamma B_0 \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -\dfrac{\gamma B_0 \hbar}{2} & 0 \\ 0 & \dfrac{\gamma B_0 \hbar}{2} \end{pmatrix} \end{aligned}$$

Determine the eigenvalues.

$$\begin{vmatrix} -\dfrac{\gamma B_0 \hbar}{2} - \lambda & 0 \\ 0 & \dfrac{\gamma B_0 \hbar}{2} - \lambda \end{vmatrix} = 0$$

$$\left(-\frac{\gamma B_0 \hbar}{2} - \lambda\right)\left(\frac{\gamma B_0 \hbar}{2} - \lambda\right) - (0)(0) = 0$$

$$\left(\lambda + \frac{\gamma B_0 \hbar}{2}\right)\left(\lambda - \frac{\gamma B_0 \hbar}{2}\right) = 0$$

$$\lambda = \pm\frac{\gamma B_0 \hbar}{2}$$

Let

$$\lambda_- = -\frac{\gamma B_0 \hbar}{2} \quad \text{and} \quad \lambda_+ = +\frac{\gamma B_0 \hbar}{2}.$$

Determine the eigenfunctions associated with these eigenvalues.

$$(\mathsf{H} - \lambda_- \mathsf{I})\mathsf{x}_- = 0 \qquad\qquad (\mathsf{H} - \lambda_+ \mathsf{I})\mathsf{x}_+ = 0$$

$$\begin{pmatrix} 0 & 0 \\ 0 & \gamma B_0 \hbar \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} -\gamma B_0 \hbar & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 0 &= 0 \\ \gamma B_0 \hbar x_2 &= 0 \end{aligned}\right\} \qquad\qquad \left.\begin{aligned} -\gamma B_0 \hbar x_1 &= 0 \\ 0 &= 0 \end{aligned}\right\}$$

$$x_2 = 0 \qquad\qquad x_1 = 0$$

$$\mathsf{x}_- = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad \mathsf{x}_+ = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The multiplicative factors, x_1 and x_2, are arbitrary but chosen so that the eigenfunctions are normalized.

$$\mathsf{x}_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi_+ \qquad\qquad \mathsf{x}_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi_-$$

x_- happens to be the spinor that represents spin-up, and x_+ happens to be the spinor that represents spin-down. Make a change in notation for the sake of consistency.

$$E_+ = \lambda_- = -\frac{\gamma B_0 \hbar}{2} \qquad\qquad E_- = \lambda_+ = +\frac{\gamma B_0 \hbar}{2}$$

These are the possible energies of the particle; negative energy is associated with spin-up, and positive energy is associated with spin-down. Because the Hamiltonian is time-independent, the Schrödinger equation,

$$i\hbar \frac{\partial \chi}{\partial t} = \mathsf{H}\chi,$$

is separable and has a general solution that is a linear combination of the stationary states multiplied by their respective wiggle factors. This is due to the principle of superposition.

$$\begin{aligned} \chi &= a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} \\ &= a\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\left(-\frac{\gamma B_0 \hbar}{2}\right)t/\hbar} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\left(\frac{\gamma B_0 \hbar}{2}\right)t/\hbar} \\ &= a\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\gamma B_0 t/2} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\gamma B_0 t/2} \\ &= \begin{pmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{pmatrix} \end{aligned}$$

a and b are arbitrary but subject to the normalization condition: $|a|^2 + |b|^2 = 1$. Use Mr. Griffiths's choice of $a = \cos(\alpha/2)$ and $b = \sin(\alpha/2)$ for convenience.

$$\chi = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2} \\ \sin\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/2} \end{pmatrix}$$

Part (a)

The probability of measuring $+\hbar/2$ for the component of spin angular momentum along the x-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_+^{(x)}\rangle$.

$$P\left(+\frac{\hbar}{2}\right) = \left|c_+^{(x)}\right|^2$$

$$= \left|\langle\chi_+^{(x)}\,|\,\chi\rangle\right|^2$$

$$= \left|\chi_+^{(x)\dagger}\chi\right|^2$$

The normalized eigenfunction (or rather eigenspinor in this context) associated with the eigenvalue $+\hbar/2$ for S_x is in Equation 4.151 on page 169.

$$\chi_+^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore,

$$P\left(+\frac{\hbar}{2}\right) = \left|\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} \\ \sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2} \end{pmatrix}\right|^2$$

$$= \frac{1}{2}\left|\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} \\ \sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2} \end{pmatrix}\right|^2$$

$$= \frac{1}{2}\left|\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} + \sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right|^2$$

$$= \frac{1}{2}\left[\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} + \sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right]\left[\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} + \sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right]^*$$

$$= \frac{1}{2}\left[\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} + \sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right]\left[\cos\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2} + \sin\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2}\right]$$

$$= \frac{1}{2}\left[\cos^2\left(\frac{\alpha}{2}\right) + \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t} + \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t} + \sin^2\left(\frac{\alpha}{2}\right)\right]$$

$$= \frac{1}{2}\left[1 + \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\left(e^{i\gamma B_0 t} + e^{-i\gamma B_0 t}\right)\right]$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{2}\sin\alpha\right)(2\cos\gamma B_0 t)\right]$$

$$= \frac{1}{2}(1 + \sin\alpha\cos\gamma B_0 t).$$

Part (b)

The probability of measuring $+\hbar/2$ for the component of spin angular momentum along the y-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_+^{(y)}\rangle$.

$$P\left(+\frac{\hbar}{2}\right) = \left|c_+^{(y)}\right|^2$$

$$= \left|\langle\chi_+^{(y)} \mid \chi\rangle\right|^2$$

$$= \left|\chi_+^{(y)\dagger}\chi\right|^2$$

The normalized eigenfunction (or rather eigenspinor in this context) associated with the eigenvalue $+\hbar/2$ for S_y is derived in part (a) of Problem 4.32.

$$\chi_+^{(y)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Therefore,

$$P\left(+\frac{\hbar}{2}\right) = \left|\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}^\dagger \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} \\ \sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2} \end{pmatrix}\right|^2$$

$$= \frac{1}{2}\left|\begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} \\ \sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2} \end{pmatrix}\right|^2$$

$$= \frac{1}{2}\left|\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} - i\sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right|^2$$

$$= \frac{1}{2}\left[\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} - i\sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right]\left[\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} - i\sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right]^*$$

$$= \frac{1}{2}\left[\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} - i\sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right]\left[\cos\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2} + i\sin\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2}\right]$$

$$= \frac{1}{2}\left[\cos^2\left(\frac{\alpha}{2}\right) + i\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t} - i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t} + \sin^2\left(\frac{\alpha}{2}\right)\right]$$

$$= \frac{1}{2}\left[1 + i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\left(e^{i\gamma B_0 t} - e^{-i\gamma B_0 t}\right)\right]$$

$$= \frac{1}{2}\left[1 + i\left(\frac{1}{2}\sin\alpha\right)(2i\sin\gamma B_0 t)\right]$$

$$= \frac{1}{2}(1 - \sin\alpha\sin\gamma B_0 t).$$

Part (c)

The probability of measuring $+\hbar/2$ for the component of spin angular momentum along the z-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_+\rangle$.

$$\begin{aligned} P\left(+\frac{\hbar}{2}\right) &= |c_+|^2 \\ &= |\langle \chi_+ \mid \chi \rangle|^2 \\ &= \left|\chi_+^\dagger \chi\right|^2 \end{aligned}$$

The normalized eigenfunction (or rather eigenspinor in this context) associated with the eigenvalue $+\hbar/2$ for S_z is in Equation 4.149 on page 169.

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore,

$$\begin{aligned} P\left(+\frac{\hbar}{2}\right) &= \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2} \\ \sin\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/2} \end{pmatrix} \right|^2 \\ &= \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2} \\ \sin\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/2} \end{pmatrix} \right|^2 \\ &= \left| \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2} \right|^2 \\ &= \left[\cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2}\right] \left[\cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2}\right]^* \\ &= \left[\cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2}\right] \left[\cos\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/2}\right] \\ &= \cos^2\left(\frac{\alpha}{2}\right). \end{aligned}$$