Problem 4.37

(a) Apply S_- to $|1\,0\rangle$ (Equation 4.175), and confirm that you get $\sqrt{2}\hbar|1\ -1\rangle$.

(b) Apply $S_\pm$ to $|0\,0\rangle$ (Equation 4.176), and confirm that you get zero.

(c) Show that $|1\,1\rangle$ and $|1\ -1\rangle$ (Equation 4.175) are eigenstates of S^2, with the appropriate eigenvalue.

Solution

Part (a)

Equation 4.175 is on page 177. Use the spin relationships in Equation 4.146 on page 168.

$$S_-|1\,0\rangle = S_-\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\right] \tag{4.175}$$

$$= \frac{1}{\sqrt{2}}\Big[S_-|\uparrow\downarrow\rangle + S_-|\downarrow\uparrow\rangle\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[(S_-^{(1)} + S_-^{(2)})|\uparrow\downarrow\rangle + (S_-^{(1)} + S_-^{(2)})|\downarrow\uparrow\rangle\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[\left(S_-^{(1)}|\uparrow\rangle\right)|\downarrow\rangle + |\uparrow\rangle\left(S_-^{(2)}|\downarrow\rangle\right) + \left(S_-^{(1)}|\downarrow\rangle\right)|\uparrow\rangle + |\downarrow\rangle\left(S_-^{(2)}|\uparrow\rangle\right)\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[(\hbar|\downarrow\rangle)|\downarrow\rangle + |\uparrow\rangle(\mathbf{0}) + (\mathbf{0})|\uparrow\rangle + |\downarrow\rangle(\hbar|\downarrow\rangle)\Big]$$

$$= \frac{1}{\sqrt{2}}(\hbar|\downarrow\downarrow\rangle + \hbar|\downarrow\downarrow\rangle)$$

$$= \frac{1}{\sqrt{2}}(2\hbar|\downarrow\downarrow\rangle)$$

$$= \sqrt{2}\hbar|\downarrow\downarrow\rangle$$

$$= \sqrt{2}\hbar|1\ -1\rangle$$

Part (b)

Equation 4.176 is on page 177 too. Use the spin relationships in Equation 4.146 on page 168.

$$\begin{aligned}
S_+|0\,0\rangle &= S_+\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right] \\
&= \frac{1}{\sqrt{2}}\left[S_+|\uparrow\downarrow\rangle - S_+|\downarrow\uparrow\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[(S_+^{(1)} + S_+^{(2)})|\uparrow\downarrow\rangle - (S_+^{(1)} + S_+^{(2)})|\downarrow\uparrow\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[\left(S_+^{(1)}|\uparrow\rangle\right)|\downarrow\rangle + |\uparrow\rangle\left(S_+^{(2)}|\downarrow\rangle\right) \right. \\
&\qquad \left. - \left(S_+^{(1)}|\downarrow\rangle\right)|\uparrow\rangle - |\downarrow\rangle\left(S_+^{(2)}|\uparrow\rangle\right)\right] \\
&= \frac{1}{\sqrt{2}}\left[(\mathbf{0})|\downarrow\rangle + |\uparrow\rangle(\hbar|\uparrow\rangle) \right. \\
&\qquad \left. - (\hbar|\uparrow\rangle)|\uparrow\rangle - |\downarrow\rangle(\mathbf{0})\right] \\
&= \frac{1}{\sqrt{2}}(\hbar|\uparrow\uparrow\rangle - \hbar|\uparrow\uparrow\rangle) \\
&= \mathbf{0}
\end{aligned}$$

$$\begin{aligned}
S_-|0\,0\rangle &= S_-\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right] \\
&= \frac{1}{\sqrt{2}}\left[S_-|\uparrow\downarrow\rangle - S_-|\downarrow\uparrow\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[(S_-^{(1)} + S_-^{(2)})|\uparrow\downarrow\rangle - (S_-^{(1)} + S_-^{(2)})|\downarrow\uparrow\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[\left(S_-^{(1)}|\uparrow\rangle\right)|\downarrow\rangle + |\uparrow\rangle\left(S_-^{(2)}|\downarrow\rangle\right) \right. \\
&\qquad \left. - \left(S_-^{(1)}|\downarrow\rangle\right)|\uparrow\rangle - |\downarrow\rangle\left(S_-^{(2)}|\uparrow\rangle\right)\right] \\
&= \frac{1}{\sqrt{2}}\left[(\hbar|\downarrow\rangle)|\downarrow\rangle + |\uparrow\rangle(\mathbf{0}) \right. \\
&\qquad \left. - (\mathbf{0})|\uparrow\rangle - |\downarrow\rangle(\hbar|\downarrow\rangle)\right] \\
&= \frac{1}{\sqrt{2}}(\hbar|\downarrow\downarrow\rangle - \hbar|\downarrow\downarrow\rangle) \\
&= \mathbf{0}
\end{aligned}$$

Part (c)

Let S^2 act on $|1\,1\rangle$ first. Use the relationships in Equation 4.171 on page 176.

$$
\begin{aligned}
S^2|1\,1\rangle &= (\mathbf{S}\cdot\mathbf{S})|\uparrow\uparrow\rangle \\
&= (\mathbf{S}^{(1)}+\mathbf{S}^{(2)})\cdot(\mathbf{S}^{(1)}+\mathbf{S}^{(2)})|\uparrow\uparrow\rangle \\
&= \left(\mathbf{S}^{(1)}\cdot\mathbf{S}^{(1)}+\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}+\mathbf{S}^{(2)}\cdot\mathbf{S}^{(1)}+\mathbf{S}^{(2)}\cdot\mathbf{S}^{(2)}\right)|\uparrow\uparrow\rangle \\
&= \left[\left(S^{(1)}\right)^2+2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}+\left(S^{(2)}\right)^2\right]|\uparrow\uparrow\rangle \\
&= \left(S^{(1)}\right)^2|\uparrow\uparrow\rangle+\left(S^{(2)}\right)^2|\uparrow\uparrow\rangle+2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}|\uparrow\uparrow\rangle \\
&= \left(S^{(1)}\right)^2\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\right\rangle+\left(S^{(2)}\right)^2\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\right\rangle+2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}|\uparrow\uparrow\rangle \\
&= \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\right\rangle+\frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\right\rangle+2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}|\uparrow\uparrow\rangle \\
&= \frac{3\hbar^2}{2}\left|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\right\rangle+2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}|\uparrow\uparrow\rangle \\
&= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle+2\left\langle S_x^{(1)},S_y^{(1)},S_z^{(1)}\right\rangle\cdot\left\langle S_x^{(2)},S_y^{(2)},S_z^{(2)}\right\rangle|\uparrow\uparrow\rangle \\
&= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle+2\left\langle \frac{1}{2}(S_+^{(1)}+S_-^{(1)}),\frac{1}{2i}(S_+^{(1)}-S_-^{(1)}),S_z^{(1)}\right\rangle\cdot\left\langle \frac{1}{2}(S_+^{(2)}+S_-^{(2)}),\frac{1}{2i}(S_+^{(2)}-S_-^{(2)}),S_z^{(2)}\right\rangle|\uparrow\uparrow\rangle \\
&= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle+\left[\frac{1}{2}(S_+^{(1)}+S_-^{(1)})(S_+^{(2)}+S_-^{(2)})-\frac{1}{2}(S_+^{(1)}-S_-^{(1)})(S_+^{(2)}-S_-^{(2)})+2S_z^{(1)}S_z^{(2)}\right]|\uparrow\uparrow\rangle \\
&= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle+\frac{1}{2}(S_+^{(1)}+S_-^{(1)})(S_+^{(2)}+S_-^{(2)})|\uparrow\uparrow\rangle-\frac{1}{2}(S_+^{(1)}-S_-^{(1)})(S_+^{(2)}-S_-^{(2)})|\uparrow\uparrow\rangle+2S_z^{(1)}S_z^{(2)}|\uparrow\uparrow\rangle \\
&= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle+\frac{1}{2}(S_+^{(1)}+S_-^{(1)})(S_+^{(2)}|\uparrow\uparrow\rangle+S_-^{(2)}|\uparrow\uparrow\rangle)-\frac{1}{2}(S_+^{(1)}-S_-^{(1)})(S_+^{(2)}|\uparrow\uparrow\rangle-S_-^{(2)}|\uparrow\uparrow\rangle) \\
&\qquad +2S_z^{(1)}\left[|\uparrow\rangle\left(S_z^{(2)}|\uparrow\rangle\right)\right] \\
&= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle+\frac{1}{2}\left(\cancel{S_+^{(1)}S_+^{(2)}|\uparrow\uparrow\rangle}+S_+^{(1)}S_-^{(2)}|\uparrow\uparrow\rangle+S_-^{(1)}S_+^{(2)}|\uparrow\uparrow\rangle+\cancel{S_-^{(1)}S_-^{(2)}|\uparrow\uparrow\rangle}\right) \\
&\qquad -\frac{1}{2}\left(\cancel{S_+^{(1)}S_+^{(2)}|\uparrow\uparrow\rangle}-S_+^{(1)}S_-^{(2)}|\uparrow\uparrow\rangle-S_-^{(1)}S_+^{(2)}|\uparrow\uparrow\rangle+\cancel{S_-^{(1)}S_-^{(2)}|\uparrow\uparrow\rangle}\right)+2S_z^{(1)}\left[|\uparrow\rangle\left(\frac{\hbar}{2}|\uparrow\rangle\right)\right]
\end{aligned}
$$

Continue the simplification.

$$\begin{aligned}
S^2|1\,1\rangle &= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle + S_+^{(1)}S_-^{(2)}|\uparrow\uparrow\rangle + S_-^{(1)}S_+^{(2)}|\uparrow\uparrow\rangle + \hbar S_z^{(1)}|\uparrow\uparrow\rangle \\
&= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle + S_+^{(1)}\left[|\uparrow\rangle\left(S_-^{(2)}|\uparrow\rangle\right)\right] + S_-^{(1)}\left[|\uparrow\rangle\left(S_+^{(2)}|\uparrow\rangle\right)\right] + \hbar\left(S_z^{(1)}|\uparrow\rangle\right)|\uparrow\rangle \\
&= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle + S_+^{(1)}\Big[|\uparrow\rangle(\hbar|\downarrow\rangle)\Big] + S_-^{(1)}\Big[|\uparrow\rangle(\mathbf{0})\Big] + \hbar\left(\frac{\hbar}{2}|\uparrow\rangle\right)|\uparrow\rangle \\
&= \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle + \hbar S_+^{(1)}|\uparrow\downarrow\rangle + \frac{\hbar^2}{2}|\uparrow\uparrow\rangle \\
&= 2\hbar^2|\uparrow\uparrow\rangle + \hbar\left(S_+^{(1)}|\uparrow\rangle\right)|\downarrow\rangle \\
&= 2\hbar^2|\uparrow\uparrow\rangle + \hbar(\mathbf{0})|\downarrow\rangle \\
&= 2\hbar^2|\uparrow\uparrow\rangle \\
&= 2\hbar^2|1\,1\rangle
\end{aligned}$$

Now let S^2 act on $|1\ {-1}\rangle$.

$$\begin{aligned}
S^2|1\ {-1}\rangle &= (\mathbf{S}\cdot\mathbf{S})|\downarrow\downarrow\rangle \\
&= (\mathbf{S}^{(1)}+\mathbf{S}^{(2)})\cdot(\mathbf{S}^{(1)}+\mathbf{S}^{(2)})|\downarrow\downarrow\rangle \\
&= \left(\mathbf{S}^{(1)}\cdot\mathbf{S}^{(1)} + \mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} + \mathbf{S}^{(2)}\cdot\mathbf{S}^{(1)} + \mathbf{S}^{(2)}\cdot\mathbf{S}^{(2)}\right)|\downarrow\downarrow\rangle \\
&= \left[\left(S^{(1)}\right)^2 + 2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} + \left(S^{(2)}\right)^2\right]|\downarrow\downarrow\rangle \\
&= \left(S^{(1)}\right)^2|\downarrow\downarrow\rangle + \left(S^{(2)}\right)^2|\downarrow\downarrow\rangle + 2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}|\downarrow\downarrow\rangle \\
&= \left(S^{(1)}\right)^2\left|\frac{1}{2}\ \frac{1}{2}\ \frac{-1}{2}\ \frac{-1}{2}\right\rangle + \left(S^{(2)}\right)^2\left|\frac{1}{2}\ \frac{1}{2}\ \frac{-1}{2}\ \frac{-1}{2}\right\rangle + 2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}|\downarrow\downarrow\rangle \\
&= \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\left|\frac{1}{2}\ \frac{1}{2}\ \frac{-1}{2}\ \frac{-1}{2}\right\rangle + \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\left|\frac{1}{2}\ \frac{1}{2}\ \frac{-1}{2}\ \frac{-1}{2}\right\rangle + 2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}|\downarrow\downarrow\rangle \\
&= \frac{3\hbar^2}{2}\left|\frac{1}{2}\ \frac{1}{2}\ \frac{-1}{2}\ \frac{-1}{2}\right\rangle + 2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}|\downarrow\downarrow\rangle \\
&= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + 2\left\langle S_x^{(1)}, S_y^{(1)}, S_z^{(1)}\right\rangle\cdot\left\langle S_x^{(2)}, S_y^{(2)}, S_z^{(2)}\right\rangle|\downarrow\downarrow\rangle
\end{aligned}$$

Continue the simplification, noting that $S_x = \frac{1}{2}(S_+ + S_-)$ and $S_y = \frac{1}{2i}(S_+ - S_-)$ (page 168).

$$\begin{aligned}
S^2|1\ -1\rangle &= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + 2\left\langle \frac{1}{2}(S_+^{(1)} + S_-^{(1)}), \frac{1}{2i}(S_+^{(1)} - S_-^{(1)}), S_z^{(1)}\right\rangle \cdot \left\langle \frac{1}{2}(S_+^{(2)} + S_-^{(2)}), \frac{1}{2i}(S_+^{(2)} - S_-^{(2)}), S_z^{(2)}\right\rangle |\downarrow\downarrow\rangle \\
&= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + \left[\frac{1}{2}(S_+^{(1)} + S_-^{(1)})(S_+^{(2)} + S_-^{(2)}) - \frac{1}{2}(S_+^{(1)} - S_-^{(1)})(S_+^{(2)} - S_-^{(2)}) + 2S_z^{(1)}S_z^{(2)}\right]|\downarrow\downarrow\rangle \\
&= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + \frac{1}{2}(S_+^{(1)} + S_-^{(1)})(S_+^{(2)} + S_-^{(2)})|\downarrow\downarrow\rangle - \frac{1}{2}(S_+^{(1)} - S_-^{(1)})(S_+^{(2)} - S_-^{(2)})|\downarrow\downarrow\rangle + 2S_z^{(1)}S_z^{(2)}|\downarrow\downarrow\rangle \\
&= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + \frac{1}{2}(S_+^{(1)} + S_-^{(1)})(S_+^{(2)}|\downarrow\downarrow\rangle + S_-^{(2)}|\downarrow\downarrow\rangle) - \frac{1}{2}(S_+^{(1)} - S_-^{(1)})(S_+^{(2)}|\downarrow\downarrow\rangle - S_-^{(2)}|\downarrow\downarrow\rangle) \\
&\qquad + 2S_z^{(1)}\left[|\downarrow\rangle\left(S_z^{(2)}|\downarrow\rangle\right)\right] \\
&= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + \frac{1}{2}\left(\cancel{S_+^{(1)}S_+^{(2)}|\downarrow\downarrow\rangle} + S_+^{(1)}S_-^{(2)}|\downarrow\downarrow\rangle + S_-^{(1)}S_+^{(2)}|\downarrow\downarrow\rangle + \cancel{S_-^{(1)}S_-^{(2)}|\downarrow\downarrow\rangle}\right) \\
&\qquad - \frac{1}{2}\left(\cancel{S_+^{(1)}S_+^{(2)}|\downarrow\downarrow\rangle} - S_+^{(1)}S_-^{(2)}|\downarrow\downarrow\rangle - S_-^{(1)}S_+^{(2)}|\downarrow\downarrow\rangle + \cancel{S_-^{(1)}S_-^{(2)}|\downarrow\downarrow\rangle}\right) + 2S_z^{(1)}\left[|\downarrow\rangle\left(-\frac{\hbar}{2}|\downarrow\rangle\right)\right] \\
&= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + S_+^{(1)}S_-^{(2)}|\downarrow\downarrow\rangle + S_-^{(1)}S_+^{(2)}|\downarrow\downarrow\rangle - \hbar S_z^{(1)}|\downarrow\downarrow\rangle \\
&= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + S_+^{(1)}\left[|\downarrow\rangle\left(S_-^{(2)}|\downarrow\rangle\right)\right] + S_-^{(1)}\left[|\downarrow\rangle\left(S_+^{(2)}|\downarrow\rangle\right)\right] - \hbar\left(S_z^{(1)}|\downarrow\rangle\right)|\downarrow\rangle \\
&= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + S_+^{(1)}\Big[|\downarrow\rangle(\mathbf{0})\Big] + S_-^{(1)}\Big[|\downarrow\rangle(\hbar|\uparrow\rangle)\Big] - \hbar\left(-\frac{\hbar}{2}|\downarrow\rangle\right)|\downarrow\rangle \\
&= \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + \hbar S_-^{(1)}|\downarrow\uparrow\rangle + \frac{\hbar^2}{2}|\downarrow\downarrow\rangle \\
&= 2\hbar^2|\downarrow\downarrow\rangle + \hbar\left(S_-^{(1)}|\downarrow\rangle\right)|\uparrow\rangle \\
&= 2\hbar^2|\downarrow\downarrow\rangle + \hbar(\mathbf{0})|\uparrow\rangle \\
&= 2\hbar^2|\downarrow\downarrow\rangle \\
&= 2\hbar^2|1\ -1\rangle
\end{aligned}$$

Therefore, $|1\,1\rangle$ and $|1\ -1\rangle$ are eigenstates of S^2, and the eigenvalue of S^2 is $2\hbar^2$.