Problem 4.41

Determine the commutator of S^2 with $S_z^{(1)}$ (where $\mathbf{S} \equiv \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$. Generalize your result to show that

$$\left[S^2, \mathbf{S}^{(1)}\right] = 2i\hbar\left(\mathbf{S}^{(1)} \times \mathbf{S}^{(2)}\right). \tag{4.185}$$

Comment: Because $S_z^{(1)}$ does not commute with S^2, we cannot hope to find states that are simultaneous eigenvectors of both. In order to form eigenstates of S^2 we need *linear combinations* of eigenstates of $S_z^{(1)}$. This is precisely what the Clebsch–Gordon coefficients (in Equation 4.183) do for us. On the other hand, it follows by obvious inference from Equation 4.185 that the *sum* $\mathbf{S}^{(1)} + \mathbf{S}^{(2)}$ *does* commute with S^2, which only confirms what we already knew (see Equation 4.103).]

[TYPO: Remove the square bracket at the end.]

Solution

Recall the fundamental commutation relations for spin angular momentum on page 166,

$$\left.\begin{aligned} [S_x, S_y] &= i\hbar S_z \\ [S_y, S_z] &= i\hbar S_x \\ [S_z, S_x] &= i\hbar S_y \end{aligned}\right\} \quad \Rightarrow \quad [S_j, S_k] = i\hbar \sum_{l=1}^{3} \varepsilon_{jkl} S_l,$$

and expand the right side.

$$\begin{aligned} 2i\hbar\left(\mathbf{S}^{(1)} \times \mathbf{S}^{(2)}\right) &= 2i\hbar\left[\sum_{j=1}^{3} \boldsymbol{\delta}_j S_j^{(1)}\right] \times \left[\sum_{k=1}^{3} \boldsymbol{\delta}_k S_k^{(2)}\right] \\ &= 2i\hbar \sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) S_j^{(1)} S_k^{(2)} \\ &= 2i\hbar \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} S_j^{(1)} S_k^{(2)} \\ &= 2i\hbar \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{klj} S_j^{(1)} S_k^{(2)} \\ &= 2\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \left[i\hbar \sum_{j=1}^{3} \varepsilon_{klj} S_j^{(1)}\right] S_k^{(2)} \\ &= 2\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \left[S_k^{(1)}, S_l^{(1)}\right] S_k^{(2)} \end{aligned}$$

Recall also from page 108 that $\left[\hat{A}\hat{B},\hat{C}\right] = \hat{A}\left[\hat{B},\hat{C}\right] + \left[\hat{A},\hat{C}\right]\hat{B}$. Since the operators corresponding to particle 1 commute with those corresponding to particle 2, $\left[S_k^{(2)}, S_l^{(1)}\right] = \mathbf{0}$.

$$\begin{aligned}
2i\hbar\left(\mathbf{S}^{(1)}\times\mathbf{S}^{(2)}\right) &= 2\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\left[S_k^{(1)}S_k^{(2)}, S_l^{(1)}\right] \\
&= 2\sum_{l=1}^{3}\boldsymbol{\delta}_l\left[\sum_{k=1}^{3}S_k^{(1)}S_k^{(2)}, S_l^{(1)}\right] \\
&= 2\sum_{l=1}^{3}\boldsymbol{\delta}_l\left[\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}, S_l^{(1)}\right] \\
&= 2\left[\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}, \sum_{l=1}^{3}\boldsymbol{\delta}_l S_l^{(1)}\right] \\
&= 2\left[\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}, \mathbf{S}^{(1)}\right] \\
&= \left[2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}, \mathbf{S}^{(1)}\right] \\
&= \underbrace{\left[\left(S^{(1)}\right)^2, \mathbf{S}^{(1)}\right]}_{\substack{=\mathbf{0} \\ \text{the square of spin angular} \\ \text{momentum commutes with spin} \\ \text{angular momentum components}}} + \left[2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}, \mathbf{S}^{(1)}\right] + \underbrace{\left[\left(S^{(2)}\right)^2, \mathbf{S}^{(1)}\right]}_{\substack{=\mathbf{0} \\ \text{the operators corresponding} \\ \text{to particle 1 commute with the} \\ \text{operators corresponding to particle 2}}} \\
&= \left[\left(S^{(1)}\right)^2, \mathbf{S}^{(1)}\right] + \left[2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}, \mathbf{S}^{(1)}\right] + \left[\left(S^{(2)}\right)^2, \mathbf{S}^{(1)}\right] \\
&= \left[\left(S^{(1)}\right)^2 + 2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} + \left(S^{(2)}\right)^2, \mathbf{S}^{(1)}\right] \\
&= \left[\left(\mathbf{S}^{(1)}\right)\cdot\left(\mathbf{S}^{(1)}\right) + \mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} + \mathbf{S}^{(2)}\cdot\mathbf{S}^{(1)} + \left(\mathbf{S}^{(2)}\right)\cdot\left(\mathbf{S}^{(2)}\right), \mathbf{S}^{(1)}\right] \\
&= \left[\left(\mathbf{S}^{(1)}+\mathbf{S}^{(2)}\right)\cdot\left(\mathbf{S}^{(1)}+\mathbf{S}^{(2)}\right), \mathbf{S}^{(1)}\right] \\
&= \left[\mathbf{S}\cdot\mathbf{S}, \mathbf{S}^{(1)}\right] \\
&= \left[S^2, \mathbf{S}^{(1)}\right].
\end{aligned}$$

Let $\boldsymbol{\delta}_x$, $\boldsymbol{\delta}_y$, and $\boldsymbol{\delta}_z$ be the unit vectors in the x-, y-, and z-directions, respectively. Take the dot product of $\boldsymbol{\delta}_z$ with both sides to get the commutator of S^2 with $S_z^{(1)}$.

$$\boldsymbol{\delta}_z \cdot \left[S^2, \mathbf{S}^{(1)}\right] = \boldsymbol{\delta}_z \cdot 2i\hbar\left(\mathbf{S}^{(1)} \times \mathbf{S}^{(2)}\right)$$

$$\left[S^2, \boldsymbol{\delta}_z \cdot \mathbf{S}^{(1)}\right] = \boldsymbol{\delta}_z \cdot 2i\hbar \begin{vmatrix} \boldsymbol{\delta}_x & \boldsymbol{\delta}_y & \boldsymbol{\delta}_z \\ S_x^{(1)} & S_y^{(1)} & S_z^{(1)} \\ S_x^{(2)} & S_y^{(2)} & S_z^{(2)} \end{vmatrix}$$

$$\left[S^2, S_z^{(1)}\right] = 2i\hbar \begin{vmatrix} 0 & 0 & 1 \\ S_x^{(1)} & S_y^{(1)} & S_z^{(1)} \\ S_x^{(2)} & S_y^{(2)} & S_z^{(2)} \end{vmatrix}$$

$$= 2i\hbar\left[S_x^{(1)} S_y^{(2)} - S_y^{(1)} S_x^{(2)}\right]$$