Problem 4.43

Suppose

$$\mathbf{A} = \frac{B_0}{2}\left(x\hat{j} - y\hat{i}\right), \quad \text{and} \quad \varphi = Kz^2,$$

where B_0 and K are constants.

(a) Find the fields $\mathbf{E}$ and $\mathbf{B}$.

(b) Find the allowed energies, for a particle of mass m and charge q, in these fields. *Answer:*

$$E(n_1, n_2) = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2, \quad (n_1, n_2 = 0, 1, 2, \ldots), \tag{4.195}$$

where $\omega_1 \equiv qB_0/m$ and $\omega_2 \equiv \sqrt{2qK/m}$. *Comment:* In two dimensions (x and y, with $K = 0$) this is the quantum analog to **cyclotron motion**; ω_1 is the classical cyclotron frequency, and ω_2 is zero. The allowed energies, $\left(n_1 + \frac{1}{2}\right)\hbar\omega_1$, are called **Landau Levels**.[60]

Solution

Part (a)

Use the definitions of the scalar and vector potential functions in Equation 4.189 on page 181 to get the fields. The electric field is

$$\begin{aligned}
\mathbf{E} &= -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} \\
&= -\left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle - \frac{\partial}{\partial t}\left(\frac{B_0}{2}\langle -y, x, 0\rangle\right) \\
&= -\left\langle \frac{\partial}{\partial x}(Kz^2), \frac{\partial}{\partial y}(Kz^2), \frac{\partial}{\partial z}(Kz^2) \right\rangle - \frac{B_0}{2}\left\langle \frac{\partial}{\partial t}(-y), \frac{\partial}{\partial t}(x), \frac{\partial}{\partial t}(0) \right\rangle \\
&= -\langle 0, 0, 2Kz\rangle - \frac{B_0}{2}\langle 0, 0, 0\rangle \\
&= \langle 0, 0, -2Kz\rangle \\
&= -2Kz\hat{\mathbf{z}},
\end{aligned}$$

and the magnetic field is

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{B_0}{2}y & \frac{B_0}{2}x & 0 \end{vmatrix} = \frac{B_0}{2}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \frac{B_0}{2}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right]\hat{\mathbf{z}} = \frac{B_0}{2}(2)\hat{\mathbf{z}}$$

$$= B_0\hat{\mathbf{z}}.$$

[60] For further discussion see Leslie E. Ballentine, *Quantum Mechanics: A Modern Development*, World Scientific, Singapore (1998), Section 11.3.

Part (b)

To find the allowed energies for a particle of mass m and charge q in these fields, solve the Schrödinger equation with the appropriate Hamiltonian using the method of separation of variables and determine the eigenvalues.

$$\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= \hat{H}\Psi \\
&= \left[\frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + q\varphi\right]\Psi \\
&= \left[\frac{1}{2m}(-i\hbar\nabla-q\mathbf{A})^2 + q\varphi\right]\Psi \\
&= \frac{1}{2m}(i\hbar\nabla+q\mathbf{A})^2\Psi + q\varphi\Psi \\
&= \frac{1}{2m}(i\hbar\nabla+q\mathbf{A})\cdot(i\hbar\nabla\Psi+q\mathbf{A}\Psi) + q\varphi\Psi \\
&= \frac{1}{2m}[-\hbar^2\nabla^2\Psi + iq\hbar[\nabla\cdot(\mathbf{A}\Psi)] + iq\hbar\mathbf{A}\cdot\nabla\Psi + q^2\mathbf{A}\cdot\mathbf{A}\Psi] + q\varphi\Psi \\
&= \frac{1}{2m}\{-\hbar^2\nabla^2\Psi + iq\hbar[\Psi\underbrace{(\nabla\cdot\mathbf{A})}_{=0} + \mathbf{A}\cdot\nabla\Psi] + iq\hbar\mathbf{A}\cdot\nabla\Psi + q^2A^2\Psi\} + q\varphi\Psi \\
&= \frac{1}{2m}\left(-\hbar^2\nabla^2\Psi + 2iq\hbar\mathbf{A}\cdot\nabla\Psi + q^2A^2\Psi\right) + q\varphi\Psi \\
&= \frac{1}{2m}\left[-\hbar^2\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right) + iq\hbar B_0\left(-y\frac{\partial\Psi}{\partial x}+x\frac{\partial\Psi}{\partial y}\right) + \frac{B_0^2q^2}{4}(x^2+y^2)\Psi\right] + q(Kz^2)\Psi
\end{aligned}$$

Make the change of variables, $x = r\cos\phi$ and $y = r\sin\phi$, and use the chain rule to write the derivatives with respect to these new variables.

$$\frac{\partial\Psi}{\partial r} = \frac{\partial\Psi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\Psi}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial\Psi}{\partial x}(\cos\phi) + \frac{\partial\Psi}{\partial y}(\sin\phi) = \frac{\partial\Psi}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial\Psi}{\partial y}\left(\frac{y}{r}\right)$$

$$\frac{\partial\Psi}{\partial \phi} = \frac{\partial\Psi}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial\Psi}{\partial y}\frac{\partial y}{\partial \phi} = \frac{\partial\Psi}{\partial x}(-r\sin\phi) + \frac{\partial\Psi}{\partial y}(r\cos\phi) = \frac{\partial\Psi}{\partial x}(-y) + \frac{\partial\Psi}{\partial y}(x)$$

Rewrite the Laplacian operator in cylindrical coordinates (r, ϕ, z) and introduce the angular frequencies, $\omega_1 = qB_0/m$ and $\omega_2 = \sqrt{2qK/m}$, and the operator $L_z = -i\hbar(\partial/\partial\phi)$ in order to make the PDE separable.

$$\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= \frac{1}{2m}\left[-\hbar^2\left(\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2}+\frac{\partial^2\Psi}{\partial z^2}\right) + iq\hbar B_0\frac{\partial\Psi}{\partial\phi} + \frac{B_0^2q^2}{4}r^2\Psi\right] + qKz^2\Psi \\
&= -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2}+\frac{\partial^2\Psi}{\partial z^2}\right) + \frac{i\hbar}{2}\omega_1\frac{\partial\Psi}{\partial\phi} + \frac{1}{8}m\omega_1^2r^2\Psi + \frac{1}{2}m\omega_2^2z^2\Psi \\
&= -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{\partial^2\Psi}{\partial z^2}\right) + \frac{1}{2mr^2}L_z^2\Psi - \frac{\omega_1}{2}L_z\Psi + \frac{1}{8}m\omega_1^2r^2\Psi + \frac{1}{2}m\omega_2^2z^2\Psi
\end{aligned}$$

When L_z acts on a wave function, the result is $\hbar m_\ell$ times that wave function. m_ℓ is the magnetic quantum number and takes on integer values: $m_\ell = 0, \pm 1, \pm 2, \ldots, \pm \ell$, where $\ell = 0, 1, 2, \ldots$.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2}\right) + \frac{1}{2mr^2}(\hbar m_\ell)^2 \Psi - \frac{\omega_1}{2}(\hbar m_\ell)\Psi + \frac{1}{8}m\omega_1^2 r^2 \Psi + \frac{1}{2}m\omega_2^2 z^2 \Psi$$

Now apply the method of separation of variables: Assume a product solution of the form $\Psi(r, z, t) = R(r)Z(z)T(t)$ and plug it into Schrödinger's equation.

$$\begin{aligned} i\hbar \frac{\partial}{\partial t}[R(r)Z(z)T(t)] &= -\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial r^2}[R(r)Z(z)T(t)] + \frac{1}{r}\frac{\partial}{\partial r}[R(r)Z(z)T(t)] + \frac{\partial^2}{\partial z^2}[R(r)Z(z)T(t)]\right\} \\ &\quad + \frac{1}{2mr^2}(\hbar m_\ell)^2[R(r)Z(z)T(t)] - \frac{\omega_1}{2}(\hbar m_\ell)[R(r)Z(z)T(t)] + \frac{1}{8}m\omega_1^2 r^2[R(r)Z(z)T(t)] + \frac{1}{2}m\omega_2^2 z^2[R(r)Z(z)T(t)] \end{aligned}$$

$$\begin{aligned} i\hbar[R(r)Z(z)T'(t)] &= -\frac{\hbar^2}{2m}\left\{[R''(r)Z(z)T(t)] + \frac{1}{r}[R'(r)Z(z)T(t)] + [R(r)Z''(z)T(t)]\right\} \\ &\quad + \frac{1}{2mr^2}(\hbar m_\ell)^2[R(r)Z(z)T(t)] - \frac{\omega_1}{2}(\hbar m_\ell)[R(r)Z(z)T(t)] + \frac{1}{8}m\omega_1^2 r^2[R(r)Z(z)T(t)] + \frac{1}{2}m\omega_2^2 z^2[R(r)Z(z)T(t)] \end{aligned}$$

Divide both sides by $R(r)Z(z)T(t)$ to separate variables.

$$\begin{aligned} i\hbar \frac{T'(t)}{T(t)} &= -\frac{\hbar^2}{2m}\left[\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{Z''(z)}{Z(z)}\right] \\ &\quad + \frac{1}{2mr^2}(\hbar m_\ell)^2 - \frac{\omega_1}{2}(\hbar m_\ell) + \frac{1}{8}m\omega_1^2 r^2 + \frac{1}{2}m\omega_2^2 z^2 \end{aligned}$$

The only way a function of t can be equal to a function of r and z is if both are equal to a constant.

$$i\hbar \frac{T'(t)}{T(t)} = -\frac{\hbar^2}{2m}\left[\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{Z''(z)}{Z(z)}\right] + \frac{1}{2mr^2}(\hbar m_\ell)^2 - \frac{\omega_1}{2}(\hbar m_\ell) + \frac{1}{8}m\omega_1^2 r^2 + \frac{1}{2}m\omega_2^2 z^2 = E$$

Bring the functions of z and any constants to the right side.

$$-\frac{\hbar^2}{2m}\left[\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)}\right] + \frac{1}{2mr^2}(\hbar m_\ell)^2 + \frac{1}{8}m\omega_1^2 r^2 = E + \frac{\omega_1}{2}(\hbar m_\ell) + \frac{\hbar^2}{2m}\frac{Z''(z)}{Z(z)} - \frac{1}{2}m\omega_2^2 z^2$$

The only way a function of r can be equal to a function of z is if both are equal to a constant.

$$-\frac{\hbar^2}{2m}\left[\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}\right]+\frac{1}{2mr^2}(\hbar m_\ell)^2+\frac{1}{8}m\omega_1^2r^2=E+\frac{\omega_1}{2}(\hbar m_\ell)+\frac{\hbar^2}{2m}\frac{Z''(z)}{Z(z)}-\frac{1}{2}m\omega_2^2z^2=F$$

As a result of using the method of separation of variables, Schrödinger's equation has reduced to three equations—one in r, one in z, and one in t.

$$\left.\begin{aligned}i\hbar\frac{T'(t)}{T(t)}&=E\\ E+\frac{\omega_1}{2}(\hbar m_\ell)+\frac{\hbar^2}{2m}\frac{Z''(z)}{Z(z)}-\frac{1}{2}m\omega_2^2z^2&=F\\ -\frac{\hbar^2}{2m}\left[\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}\right]+\frac{1}{2mr^2}(\hbar m_\ell)^2+\frac{1}{8}m\omega_1^2r^2&=F\end{aligned}\right\}$$

The strategy to find E, the allowed energies, is to solve the last equation for F and then to solve the second equation for E.

$$-\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{2mr^2}(\hbar m_\ell)^2R(r)+\frac{1}{8}m\omega_1^2r^2R(r)=FR(r)$$

Make the substitution,

$$R(r)=\frac{u(r)}{\sqrt{r}}\quad\Rightarrow\quad\frac{dR}{dr}=\frac{2ru'-u}{2r^{3/2}}\quad\Rightarrow\quad\frac{d^2R}{dr^2}=\frac{4r^2u''-4ru'+3u}{4r^{5/2}}.$$

As a result, the previous equation becomes

$$-\frac{\hbar^2}{2m}\left(\frac{4r^2u''-4ru'+3u}{4r^{5/2}}+\frac{2ru'-u}{2r^{5/2}}\right)+\frac{1}{2mr^2}(\hbar m_\ell)^2\frac{u(r)}{\sqrt{r}}+\frac{1}{8}m\omega_1^2r^2\frac{u(r)}{\sqrt{r}}=F\frac{u(r)}{\sqrt{r}}$$

$$-\frac{\hbar^2}{2m}\left[\frac{d^2u}{dr^2}+\frac{1}{4r^2}u(r)\right]+\frac{1}{2mr^2}(\hbar m_\ell)^2u(r)+\frac{1}{8}m\omega_1^2r^2u(r)=Fu(r)$$

$$\frac{d^2u}{dr^2}+\frac{1}{4r^2}u(r)-\frac{m_\ell^2}{r^2}u(r)-\frac{m^2\omega_1^2}{4\hbar^2}r^2u(r)=-\frac{2mF}{\hbar^2}u(r)$$

$$\frac{d^2u}{dr^2}-\left(\frac{m^2\omega_1^2}{4\hbar^2}r^2-\frac{2mF}{\hbar^2}\right)u(r)-\frac{m_\ell^2-\frac{1}{4}}{r^2}u(r)=0.$$

This is equation (1) from page 2 of Problem 4.47 but with $\omega_1^2/4$ in place of ω^2 and $m_\ell^2-1/4$ in place of $\ell(\ell+1)$. Use the result on the bottom of page 7.

$$\begin{aligned}F&=\hbar\left(\frac{\omega_1}{2}\right)\left[2N+\left(|m_\ell|-\frac{1}{2}\right)+\frac{3}{2}\right],\quad N=0,1,2,\ldots\\ &=\frac{\hbar\omega_1}{2}\left(2N+1+|m_\ell|\right)\end{aligned}$$

Now that F is known, the equation in z can be considered.

$$E + \frac{\omega_1}{2}(\hbar m_\ell) + \frac{\hbar^2}{2m}\frac{Z''(z)}{Z(z)} - \frac{1}{2}m\omega_2^2 z^2 = F$$

$$-\frac{\hbar^2}{2m}\frac{d^2Z}{dz^2} + \frac{1}{2}m\omega_2^2 z^2 Z(z) = \left[E + \frac{\omega_1}{2}(\hbar m_\ell) - F\right] Z(z)$$

This is the TISE for a one-dimensional harmonic oscillator with angular frequency ω_2.

$$E + \frac{\omega_1}{2}(\hbar m_\ell) - F = \left(n_2 + \frac{1}{2}\right)\hbar\omega_2, \quad n_2 = 0, 1, 2, \ldots$$

Solve for E.

$$\begin{aligned} E &= F - \frac{\omega_1}{2}(\hbar m_\ell) + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2 \\ &= \frac{\hbar\omega_1}{2}(2N + 1 + |m_\ell|) - \frac{\omega_1}{2}(\hbar m_\ell) + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2 \\ &= \left(N + \frac{1}{2} + \frac{|m_\ell| - m_\ell}{2}\right)\hbar\omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2 \end{aligned}$$

This quantity, $(|m_\ell| - m_\ell)/2$, is a nonnegative integer $(0, 1, 2, \ldots, \ell)$ for any value of m_ℓ. Therefore, letting $n_1 = N + (|m_\ell| - m_\ell)/2$, the allowed energies are

$$E_{n_1 n_2} = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2, \quad \begin{cases} n_1 = 0, 1, 2, \ldots \\ n_2 = 0, 1, 2, \ldots \end{cases},$$

where $\omega_1 = qB_0/m$ and $\omega_2 = \sqrt{2qK/m}$.