Problem 4.45

(a) Derive Equation 4.199 from Equation 4.190.

(b) Derive Equation 4.211, starting with Equation 4.210.

Solution

Part (a)

Equation 4.190 is on page 181.

$$\hat{H} = \frac{1}{2m}(-i\hbar\nabla - q\mathbf{A})^2 + q\varphi \tag{4.190}$$

Apply this operator to a test function f and set $\varphi = 0$.

$$\begin{aligned}
\hat{H}f &= \frac{1}{2m}(-i\hbar\nabla - q\mathbf{A})^2 f \\
&= \frac{1}{2m}(-i\hbar\nabla - q\mathbf{A}) \cdot (-i\hbar\nabla - q\mathbf{A}) f \\
&= \frac{1}{2m}(-i\hbar\nabla - q\mathbf{A}) \cdot (-i\hbar\nabla f - q\mathbf{A} f) \\
&= \frac{1}{2m}[-i\hbar\nabla \cdot (-i\hbar\nabla f - q\mathbf{A}f) - q\mathbf{A} \cdot (-i\hbar\nabla f - q\mathbf{A}f)] \\
&= \frac{1}{2m}[i^2\hbar^2(\nabla \cdot \nabla f) + i\hbar q\nabla \cdot (\mathbf{A}f) + i\hbar q\mathbf{A} \cdot (\nabla f) + q^2(\mathbf{A} \cdot \mathbf{A})f] \\
&= \frac{1}{2m}\{-\hbar^2\nabla^2 f + i\hbar q[f(\underbrace{\nabla \cdot \mathbf{A}}_{=\,0}) + \mathbf{A} \cdot (\nabla f)] + i\hbar q\mathbf{A} \cdot (\nabla f) + q^2A^2 f\} \\
&= \frac{1}{2m}[-\hbar^2\nabla^2 f + q^2A^2 f + 2i\hbar q\mathbf{A} \cdot (\nabla f)] \\
&= \frac{1}{2m}(-\hbar^2\nabla^2 + q^2A^2 + 2i\hbar q\mathbf{A} \cdot \nabla) f
\end{aligned}$$

Therefore,

$$\hat{H} = \frac{1}{2m}\left[-\hbar^2\nabla^2 + q^2A^2 + 2i\hbar q\mathbf{A} \cdot \nabla\right]. \tag{4.199}$$

Part (b)

Equation 4.210 is on page 185.

$$(-i\hbar\nabla - q\mathbf{A})\Psi = -i\hbar e^{ig}\nabla\Psi' \tag{4.210}$$

Therefore, [noting that $\nabla g = (q/\hbar)\mathbf{A}$ and $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$]

$$\begin{aligned}
(-i\hbar\nabla - q\mathbf{A})^2\Psi &= (-i\hbar\nabla - q\mathbf{A}) \cdot [(-i\hbar\nabla - q\mathbf{A})\Psi] \\
&= (-i\hbar\nabla - q\mathbf{A}) \cdot (-i\hbar e^{ig}\nabla\Psi') \\
&= -i\hbar\nabla \cdot (-i\hbar e^{ig}\nabla\Psi') - q\mathbf{A} \cdot (-i\hbar e^{ig}\nabla\Psi') \\
&= i^2\hbar^2\nabla \cdot (e^{ig}\nabla\Psi') + i\hbar q e^{ig}\mathbf{A} \cdot (\nabla\Psi') \\
&= -\hbar^2[e^{ig}(\nabla \cdot \nabla\Psi') + \nabla\Psi' \cdot (\nabla e^{ig})] + i\hbar q e^{ig}\mathbf{A} \cdot (\nabla\Psi') \\
&= -\hbar^2\{e^{ig}(\nabla^2\Psi') + \nabla\Psi' \cdot [e^{ig} \cdot \nabla(ig)]\} + i\hbar q e^{ig}\mathbf{A} \cdot (\nabla\Psi') \\
&= -\hbar^2[e^{ig}(\nabla^2\Psi') + \nabla\Psi' \cdot (e^{ig} \cdot i\nabla g)] + i\hbar q e^{ig}\mathbf{A} \cdot (\nabla\Psi') \\
&= -\hbar^2 e^{ig}\nabla^2\Psi' - i\hbar^2 e^{ig}\nabla\Psi' \cdot \nabla g + i\hbar q e^{ig}\mathbf{A} \cdot (\nabla\Psi') \\
&= -\hbar^2 e^{ig}\nabla^2\Psi' - i\hbar^2 e^{ig}\nabla\Psi' \cdot \left(\frac{q}{\hbar}\mathbf{A}\right) + i\hbar q e^{ig}\mathbf{A} \cdot (\nabla\Psi') \\
&= -\hbar^2 e^{ig}\nabla^2\Psi' - i\hbar q e^{ig}\nabla\Psi' \cdot \mathbf{A} + i\hbar q e^{ig}\mathbf{A} \cdot (\nabla\Psi') \\
&= -\hbar^2 e^{ig}\nabla^2\Psi' - \cancel{i\hbar q e^{ig}\mathbf{A} \cdot (\nabla\Psi')} + \cancel{i\hbar q e^{ig}\mathbf{A} \cdot (\nabla\Psi')} \\
&= -\hbar^2 e^{ig}\nabla^2\Psi'.
\end{aligned} \tag{4.211}$$