Problem 4.50

The (time-independent) **momentum space wave function** in three dimensions is defined by the natural generalization of Equation 3.54:

$$\phi(\mathbf{p}) \equiv \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-i(\mathbf{p}\cdot\mathbf{r})/\hbar}\psi(\mathbf{r})\, d^3\mathbf{r}. \tag{4.223}$$

(a) Find the momentum space wave function for the ground state of hydrogen (Equation 4.80). *Hint*: Use spherical coordinates, setting the polar axis along the direction of **p**. Do the θ integral first. *Answer*:

$$\phi(\mathbf{p}) = \frac{1}{\pi}\left(\frac{2a}{\hbar}\right)^{3/2} \frac{1}{[1+(ap/\hbar)^2]^2}. \tag{4.224}$$

(b) Check that $\phi(\mathbf{p})$ is normalized.

(c) Use $\phi(\mathbf{p})$ to calculate $\langle p^2 \rangle$, in the ground state of hydrogen.

(d) What is the expectation value of the kinetic energy in this state? Express your answer as a multiple of E_1, and check that it is consistent with the virial theorem (Equation 4.218).

[Keep in mind that ϕ sloppily represents the time-independent momentum space wave function and the azimuthal angle in spherical coordinates (r, θ, ϕ).]

Solution

Part (a)

The ground state of hydrogen is in Equation 4.80 on page 148.

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

Determine the corresponding time-independent momentum space wave function by taking the triple Fourier transform of $\psi_{100}(r,\theta,\phi)$. Let β be the angle between the vectors, $\mathbf{p} = \langle p_x, p_y, p_z \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$.

$$\varphi(p_x,p_y,p_z) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint_{\text{all space}} e^{-i(\mathbf{p}\cdot\mathbf{r})/\hbar}\psi(x,y,z)\, d\mathcal{V}$$

$$\begin{aligned}
\varphi_{100}(p_x,p_y,p_z) &= \frac{1}{(2\pi\hbar)^{3/2}} \int_0^\pi \int_0^{2\pi} \int_0^\infty e^{-i(pr\cos\beta)/\hbar}\psi_{100}(r,\theta,\phi)(r^2 \sin\theta\, dr\, d\phi\, d\theta) \\
&= \frac{1}{(2\pi\hbar)^{3/2}} \int_0^\pi \int_0^{2\pi} \int_0^\infty e^{-i(pr\cos\beta)/\hbar}\left(\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}\right)(r^2 \sin\theta\, dr\, d\phi\, d\theta) \\
&= \frac{1}{2\pi^2\sqrt{2\hbar^3 a_0^3}} \int_0^\pi \int_0^{2\pi} \int_0^\infty r^2 e^{-r/a_0} e^{-i(pr\cos\beta)/\hbar} \sin\theta\, dr\, d\phi\, d\theta
\end{aligned}$$

Since $\psi_{100}(r,\theta,\phi)$ is only a function of r, the polar z-axis can be oriented in the direction of $\mathbf{p}$ to make $\beta = \theta$ without affecting the value of the integral.

$$\begin{aligned}
\varphi_{100}(p_x,p_y,p_z) &= \frac{1}{2\pi^2\sqrt{2\hbar^3 a_0^3}} \int_0^\pi \int_0^{2\pi} \int_0^\infty r^2 e^{-r/a_0} e^{-i(pr\cos\theta)/\hbar} \sin\theta\, dr\, d\phi\, d\theta \\
&= \frac{1}{2\pi^2\sqrt{2\hbar^3 a_0^3}} \left(\int_0^{2\pi} d\phi\right) \int_0^\infty r^2 e^{-r/a_0} \left[\int_0^\pi e^{-i(pr\cos\theta)/\hbar} \sin\theta\, d\theta\right] dr \\
&= \frac{1}{2\pi^2\sqrt{2\hbar^3 a_0^3}} (2\pi) \int_0^\infty r^2 e^{-r/a_0} \left[\int_{\cos 0}^{\cos\pi} e^{-ipru/\hbar}(-du)\right] dr \\
&= \frac{1}{\pi\sqrt{2\hbar^3 a_0^3}} \int_0^\infty r^2 e^{-r/a_0} \left[\int_1^{-1} e^{-ipru/\hbar}(-du)\right] dr \\
&= \frac{1}{\pi\sqrt{2\hbar^3 a_0^3}} \int_0^\infty r^2 e^{-r/a_0} \left(\int_{-1}^{1} e^{-ipru/\hbar}\, du\right) dr \\
&= \frac{1}{\pi\sqrt{2\hbar^3 a_0^3}} \int_0^\infty r^2 e^{-r/a_0} \left(-\frac{\hbar}{ipr} e^{-ipru/\hbar}\Big|_{-1}^{1}\right) dr \\
&= \frac{1}{\pi\sqrt{2\hbar^3 a_0^3}} \int_0^\infty r^2 e^{-r/a_0} \left[\frac{i\hbar}{pr}(e^{-ipr/\hbar} - e^{ipr/\hbar})\right] dr \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}} \int_0^\infty r e^{-r/a_0} (e^{-ipr/\hbar} - e^{ipr/\hbar})\, dr \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}} \left\{\int_0^\infty r \exp\left[-\left(\frac{1}{a_0} + \frac{ip}{\hbar}\right) r\right] dr - \int_0^\infty r \exp\left[-\left(\frac{1}{a_0} - \frac{ip}{\hbar}\right) r\right] dr\right\} \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}} \left\{\int_0^\infty \left[-\frac{\partial}{\partial w}(e^{-wr})\Big|_{w=\frac{1}{a_0}+\frac{ip}{\hbar}}\right] dr - \int_0^\infty \left[-\frac{\partial}{\partial w}(e^{-wr})\Big|_{w=\frac{1}{a_0}-\frac{ip}{\hbar}}\right] dr\right\} \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}} \left[-\frac{d}{dw}\left(\int_0^\infty e^{-wr}\, dr\right)\Big|_{w=\frac{1}{a_0}+\frac{ip}{\hbar}} + \frac{d}{dw}\left(\int_0^\infty e^{-wr}\, dr\right)\Big|_{w=\frac{1}{a_0}-\frac{ip}{\hbar}}\right] \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}} \left[-\frac{d}{dw}\left(\frac{1}{w}\right)\Big|_{w=\frac{1}{a_0}+\frac{ip}{\hbar}} + \frac{d}{dw}\left(\frac{1}{w}\right)\Big|_{w=\frac{1}{a_0}-\frac{ip}{\hbar}}\right] \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}} \left[-\left(-\frac{1}{w^2}\right)\Big|_{w=\frac{1}{a_0}+\frac{ip}{\hbar}} + \left(-\frac{1}{w^2}\right)\Big|_{w=\frac{1}{a_0}-\frac{ip}{\hbar}}\right] \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}} \left[\frac{1}{\left(\frac{1}{a_0} + \frac{ip}{\hbar}\right)^2} - \frac{1}{\left(\frac{1}{a_0} - \frac{ip}{\hbar}\right)^2}\right] \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}} \left[\frac{\left(\frac{1}{a_0} - \frac{ip}{\hbar}\right)^2 - \left(\frac{1}{a_0} + \frac{ip}{\hbar}\right)^2}{\left(\frac{1}{a_0} + \frac{ip}{\hbar}\right)^2 \left(\frac{1}{a_0} - \frac{ip}{\hbar}\right)^2}\right]
\end{aligned}$$

Simplify the result.

$$\begin{aligned}
\varphi_{100}(p_x,p_y,p_z) &= \frac{i}{\pi p\sqrt{2\hbar a_0^3}}\left[\frac{\left(\frac{1}{a_0^2}-\frac{2ip}{\hbar a_0}-\frac{p^2}{\hbar^2}\right)-\left(\frac{1}{a_0^2}+\frac{2ip}{\hbar a_0}-\frac{p^2}{\hbar^2}\right)}{\left(\frac{1}{a_0^2}+\frac{p^2}{\hbar^2}\right)^2}\right] \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}}\left[\frac{-\frac{4ip}{\hbar a_0}}{\left(\frac{1}{a_0^2}+\frac{p^2}{\hbar^2}\right)^2}\right] \\
&= \frac{i}{\pi p\sqrt{2\hbar a_0^3}}\left[\frac{-4ip}{\hbar a_0\left(\frac{1}{a_0^2}+\frac{p^2}{\hbar^2}\right)^2}\right] \\
&= \frac{4}{\pi\sqrt{2\hbar^3 a_0^5}}\frac{1}{\left(\frac{1}{a_0^2}+\frac{p^2}{\hbar^2}\right)^2} \\
&= \frac{4}{\pi\sqrt{2\hbar^3 a_0^5}}\frac{1}{\left[\frac{1}{a_0^2}\left(1+\frac{a_0^2p^2}{\hbar^2}\right)\right]^2} \\
&= \frac{4}{\pi\sqrt{2\hbar^3 a_0^5}}\frac{1}{\frac{1}{a_0^4}\left(1+\frac{a_0^2p^2}{\hbar^2}\right)^2} \\
&= \frac{1}{\pi}\sqrt{\frac{16}{\frac{2\hbar^3a_0^5}{a_0^8}}}\frac{1}{\left(1+\frac{a_0^2p^2}{\hbar^2}\right)^2} \\
&= \frac{1}{\pi}\sqrt{\frac{8a_0^3}{\hbar^3}}\frac{1}{\left(1+\frac{a_0^2p^2}{\hbar^2}\right)^2} \\
&= \frac{1}{\pi}\left(\frac{2a_0}{\hbar}\right)^{3/2}\frac{1}{\left[1+\left(\frac{a_0p}{\hbar}\right)^2\right]^2}
\end{aligned}$$

Part (b)

Check to see that the time-independent momentum space wave function is normalized by integrating over all of momentum space and seeing if the result is 1.

$$\begin{aligned}
\iiint_{\text{all m-space}}|\varphi_{100}|^2\,d\mathcal{V} &= \int_0^\pi\int_0^{2\pi}\int_0^\infty \frac{1}{\pi^2}\frac{8a_0^3}{\hbar^3}\frac{1}{\left(1+\frac{a_0^2p^2}{\hbar^2}\right)^4}(p^2\sin\theta\,dp\,d\phi\,d\theta) \\
&= \frac{8a_0^3}{\pi^2\hbar^3}\left(\int_0^\pi\sin\theta\,d\theta\right)\left(\int_0^{2\pi}d\phi\right)\int_0^\infty\frac{p^2}{\left(1+\frac{a_0^2p^2}{\hbar^2}\right)^4}\,dp
\end{aligned}$$

Evaluate the integrals in $d\theta$ and $d\phi$ and simplify the last integrand.

$$\iiint_{\text{all m-space}} |\varphi_{100}|^2 \, d\mathcal{V} = \frac{8a_0^3}{\pi^2\hbar^3}(2)(2\pi)\int_0^\infty \frac{p^2}{\left[\frac{a_0^2}{\hbar^2}\left(\frac{\hbar^2}{a_0^2}+p^2\right)\right]^4}\,dp$$

$$= \frac{32a_0^3}{\pi\hbar^3}\int_0^\infty \left(\frac{\hbar^8}{a_0^8}\right)\frac{p^2}{\left(p^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dp$$

$$= \frac{32\hbar^5}{\pi a_0^5}\int_0^\infty \frac{p^2}{\left(p^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dp$$

$$= \frac{16\hbar^5}{\pi a_0^5}\int_{-\infty}^\infty \frac{p^2}{\left(p^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dp \tag{1}$$

In order to evaluate this integral, consider the corresponding integral in the complex plane,

$$\oint_C \frac{z^2}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz,$$

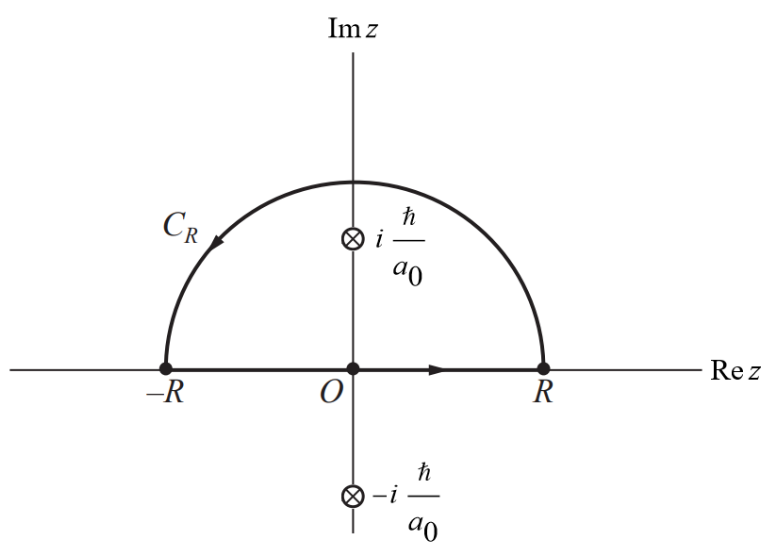
whose integrand has singularities (fourth-order poles) at

$$\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4 = 0$$

$$\left(z+i\frac{\hbar}{a_0}\right)^4\left(z-i\frac{\hbar}{a_0}\right)^4 = 0$$

$$z = \pm i\frac{\hbar}{a_0},$$

where C is the positively oriented closed loop shown below.



According to Cauchy's residue theorem, the integral of a function with k singularities within a positively oriented closed loop is equal to $2\pi i$ times the sum of the residues at these enclosed singularities.

$$\oint_C \frac{z^2}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} \frac{z^2}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}$$

C consists of the semicircular arc C_R above the real axis and the line segment on the real axis from $z=-R$ to $z=R$. Only the singularity at $z = i\frac{\hbar}{a_0}$ lies within the loop.

$$\begin{aligned}
\int_{-R}^{R} \frac{z^2}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz + \int_{C_R} \frac{z^2}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz &= 2\pi i \operatorname*{Res}_{z=i\hbar/a_0} \frac{z^2}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4} \\
&= 2\pi i \operatorname*{Res}_{z=i\hbar/a_0} \frac{z^2}{\left(z+i\frac{\hbar}{a_0}\right)^4\left(z-i\frac{\hbar}{a_0}\right)^4} \\
&= 2\pi i \operatorname*{Res}_{z=i\hbar/a_0} \frac{\frac{z^2}{\left(z+i\frac{\hbar}{a_0}\right)^4}}{\left(z-i\frac{\hbar}{a_0}\right)^4} \\
&= \frac{2\pi i}{(4-1)!}\frac{d^{4-1}}{dz^{4-1}}\left[\frac{z^2}{\left(z+i\frac{\hbar}{a_0}\right)^4}\right]\Bigg|_{z=i\hbar/a_0} \\
&= \frac{2\pi i}{6}\frac{d^2}{dz^2}\left[-\frac{2z\left(z-i\frac{\hbar}{a_0}\right)}{\left(z+i\frac{\hbar}{a_0}\right)^5}\right]\Bigg|_{z=i\hbar/a_0} \\
&= \frac{\pi i}{3}\frac{d}{dz}\left[\frac{2\left(3z^2-6i\frac{\hbar}{a_0}z-\frac{\hbar^2}{a_0^2}\right)}{\left(z+i\frac{\hbar}{a_0}\right)^6}\right]\Bigg|_{z=i\hbar/a_0} \\
&= \frac{\pi i}{3}\left[-\frac{24\left(z^2-3i\frac{\hbar}{a_0}z-\frac{\hbar^2}{a_0^2}\right)}{\left(z+i\frac{\hbar}{a_0}\right)^7}\right]\Bigg|_{z=i\hbar/a_0} \\
&= -8\pi i\left[\frac{\left(-\frac{\hbar^2}{a_0^2}+3\frac{\hbar^2}{a_0^2}-\frac{\hbar^2}{a_0^2}\right)}{\left(i\frac{2\hbar}{a_0}\right)^7}\right] \\
&= -8\pi i\left(\frac{\frac{\hbar^2}{a_0^2}}{-128i\frac{\hbar^7}{a_0^7}}\right) \\
&= \frac{\pi a_0^5}{16\hbar^5} \qquad (2)
\end{aligned}$$

Parameterize the semicircular arc C_R.

$$C_R: \quad z = Re^{i\theta}, \quad \theta = 0 \quad \rightarrow \quad \theta = \pi$$

As a result, equation (2) becomes

$$\int_{-R}^{R} \frac{z^2}{\left(z^2 + \frac{\hbar^2}{a_0^2}\right)^4}\, dz + \int_0^\pi \frac{(Re^{i\theta})^2}{\left[(Re^{i\theta})^2 + \frac{\hbar^2}{a_0^2}\right]^4} (iRe^{i\theta}\, d\theta) = \frac{\pi a_0^5}{16\hbar^5}.$$

Consider the magnitude of this second integral on the left.

$$\begin{aligned}
\left| \int_0^\pi \frac{(Re^{i\theta})^2}{\left[(Re^{i\theta})^2 + \frac{\hbar^2}{a_0^2}\right]^4} (iRe^{i\theta}\, d\theta) \right| &\leq \int_0^\pi \left| \frac{(Re^{i\theta})^2}{\left[(Re^{i\theta})^2 + \frac{\hbar^2}{a_0^2}\right]^4} (iRe^{i\theta}) \right| d\theta \\
&= \int_0^\pi \frac{|Re^{i\theta}|^2}{\left|(Re^{i\theta})^2 + \frac{\hbar^2}{a_0^2}\right|^4} |iRe^{i\theta}|\, d\theta \\
&= \int_0^\pi \frac{R^2}{\left|(Re^{i\theta})^2 + \frac{\hbar^2}{a_0^2}\right|^4} (R)\, d\theta \\
&\leq \int_0^\pi \frac{R^2}{\left||Re^{i\theta}|^2 - \frac{\hbar^2}{a_0^2}\right|^4} (R)\, d\theta \\
&= \int_0^\pi \frac{R^2}{\left(R^2 - \frac{\hbar^2}{a_0^2}\right)^4} (R)\, d\theta \\
&= \frac{R^2}{\left(R^2 - \frac{\hbar^2}{a_0^2}\right)^4} (\pi R)
\end{aligned}$$

Notice that the upper bound of this integral's magnitude tends to zero in the limit as $R \rightarrow \infty$.

$$\lim_{R\rightarrow\infty} \frac{R^2}{\left(R^2 - \frac{\hbar^2}{a_0^2}\right)^4} (\pi R) = \lim_{R\rightarrow\infty} \frac{\pi R^3}{R^8 \left(1 - \frac{\hbar^2}{a_0^2 R^2}\right)^4} = \lim_{R\rightarrow\infty} \frac{\frac{\pi}{R^5}}{\left(1 - \frac{\hbar^2}{a_0^2 R^2}\right)^4} = \frac{0}{1} = 0$$

In taking the limit of both sides of equation (2) as $R \rightarrow \infty$, then, the integral over C_R vanishes,

$$\lim_{R\rightarrow\infty} \int_{-R}^{R} \frac{z^2}{\left(z^2 + \frac{\hbar^2}{a_0^2}\right)^4}\, dz + \underbrace{\lim_{R\rightarrow\infty} \int_{C_R} \frac{z^2}{\left(z^2 + \frac{\hbar^2}{a_0^2}\right)^4}\, dz}_{=\,0} = \lim_{R\rightarrow\infty} \frac{\pi a_0^5}{16\hbar^5},$$

which means

$$\int_{-\infty}^{\infty} \frac{z^2}{\left(z^2 + \frac{\hbar^2}{a_0^2}\right)^4}\, dz = \frac{\pi a_0^5}{16\hbar^5}.$$

Equation (1) then becomes

$$\iiint_{\text{all m-space}} |\varphi_{100}|^2 \, d\mathcal{V} = \frac{16\hbar^5}{\pi a_0^5} \int_{-\infty}^{\infty} \frac{p^2}{\left(p^2 + \frac{\hbar^2}{a_0^2}\right)^4} \, dp$$

$$= \frac{16\hbar^5}{\pi a_0^5} \left(\frac{\pi a_0^5}{16\hbar^5}\right)$$

$$= 1,$$

which proves that the time-independent momentum space wave function for the ground state of hydrogen is normalized.

Part (c)

Calculate the expectation value of p^2 for the ground state of hydrogen using φ_{100}.

$$\langle p^2 \rangle = \langle \varphi_{100} \,|\, \hat{p}^2 \,|\, \varphi_{100} \rangle = \iiint_{\text{all m-space}} \varphi_{100}^* (p^2) \varphi_{100} \, d\mathcal{V}$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} \left[\frac{1}{\pi} \sqrt{\frac{8a_0^3}{\hbar^3}} \frac{1}{\left(1 + \frac{a_0^2 p^2}{\hbar^2}\right)^2} \right]^* p^2 \left[\frac{1}{\pi} \sqrt{\frac{8a_0^3}{\hbar^3}} \frac{1}{\left(1 + \frac{a_0^2 p^2}{\hbar^2}\right)^2} \right] (p^2 \sin\theta \, dp \, d\phi \, d\theta)$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} \left[\frac{1}{\pi} \sqrt{\frac{8a_0^3}{\hbar^3}} \frac{1}{\left(1 + \frac{a_0^2 p^2}{\hbar^2}\right)^2} \right] p^2 \left[\frac{1}{\pi} \sqrt{\frac{8a_0^3}{\hbar^3}} \frac{1}{\left(1 + \frac{a_0^2 p^2}{\hbar^2}\right)^2} \right] (p^2 \sin\theta \, dp \, d\phi \, d\theta)$$

$$= \frac{8a_0^3}{\pi^2 \hbar^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} \frac{p^4 \sin\theta}{\left(1 + \frac{a_0^2 p^2}{\hbar^2}\right)^4} \, dp \, d\phi \, d\theta$$

$$= \frac{8a_0^3}{\pi^2 \hbar^3} \left(\int_0^{\pi} \sin\theta \, d\theta\right) \left(\int_0^{2\pi} d\phi\right) \int_0^{\infty} \frac{p^4}{\left(1 + \frac{a_0^2 p^2}{\hbar^2}\right)^4} \, dp$$

$$= \frac{8a_0^3}{\pi^2 \hbar^3} (2)(2\pi) \int_0^{\infty} \frac{p^4}{\left[\frac{a_0^2}{\hbar^2}\left(\frac{\hbar^2}{a_0^2} + p^2\right)\right]^4} \, dp$$

$$= \frac{32a_0^3}{\pi \hbar^3} \int_0^{\infty} \left(\frac{\hbar^8}{a_0^8}\right) \frac{p^4}{\left(p^2 + \frac{\hbar^2}{a_0^2}\right)^4} \, dp$$

$$= \frac{32\hbar^5}{\pi a_0^5} \int_0^{\infty} \frac{p^4}{\left(p^2 + \frac{\hbar^2}{a_0^2}\right)^4} \, dp$$

$$= \frac{16\hbar^5}{\pi a_0^5} \int_{-\infty}^{\infty} \frac{p^4}{\left(p^2 + \frac{\hbar^2}{a_0^2}\right)^4} \, dp \tag{3}$$

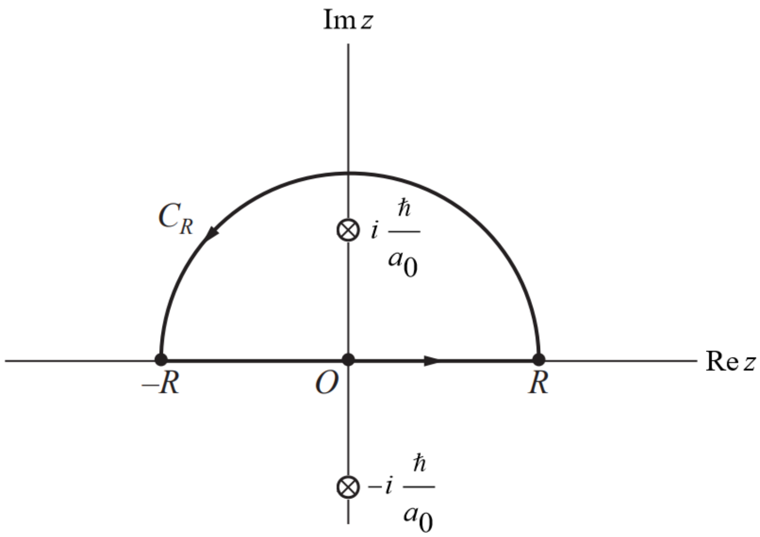
In order to evaluate this integral, consider the corresponding integral in the complex plane,

$$\oint_C \frac{z^4}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz,$$

whose integrand has singularities (fourth-order poles) at

$$\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4=0 \quad\to\quad \left(z+i\frac{\hbar}{a_0}\right)^4\left(z-i\frac{\hbar}{a_0}\right)^4=0 \quad\to\quad z=\pm i\frac{\hbar}{a_0},$$

where C is the positively oriented closed loop shown below.



According to Cauchy's residue theorem, the integral of a function with k singularities within a positively oriented closed loop is equal to $2\pi i$ times the sum of the residues at these enclosed singularities.

$$\oint_C \frac{z^4}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz = 2\pi i\sum_{j=1}^{k}\operatorname*{Res}_{z=z_j}\frac{z^4}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}$$

C consists of the semicircular arc C_R above the real axis and the line segment on the real axis from $z=-R$ to $z=R$. Only the singularity at $z=i\frac{\hbar}{a_0}$ lies within the loop.

$$\begin{aligned}\int_{-R}^{R}\frac{z^4}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz+\int_{C_R}\frac{z^4}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz &= 2\pi i\operatorname*{Res}_{z=i\hbar/a_0}\frac{z^4}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\\ &= 2\pi i\operatorname*{Res}_{z=i\hbar/a_0}\frac{z^4}{\left(z+i\frac{\hbar}{a_0}\right)^4\left(z-i\frac{\hbar}{a_0}\right)^4}\\ &= 2\pi i\operatorname*{Res}_{z=i\hbar/a_0}\frac{\frac{z^4}{\left(z+i\frac{\hbar}{a_0}\right)^4}}{\left(z-i\frac{\hbar}{a_0}\right)^4}\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\int_{-R}^{R} \frac{z^4}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz + \int_{C_R} \frac{z^4}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz &= \frac{2\pi i}{(4-1)!}\frac{d^{4-1}}{dz^{4-1}}\left[\frac{z^4}{\left(z+i\frac{\hbar}{a_0}\right)^4}\right]\Bigg|_{z=i\hbar/a_0} \\
&= \frac{2\pi i}{6}\frac{d^2}{dz^2}\left[\frac{4\left(i\frac{\hbar}{a_0}\right)z^3}{\left(z+i\frac{\hbar}{a_0}\right)^5}\right]\Bigg|_{z=i\hbar/a_0} \\
&= \frac{\pi i}{3}\frac{d}{dz}\left[-\frac{4i\frac{\hbar}{a_0}\left(2z-3i\frac{\hbar}{a_0}\right)z^2}{\left(z+i\frac{\hbar}{a_0}\right)^6}\right]\Bigg|_{z=i\hbar/a_0} \\
&= \frac{\pi i}{3}\left[\frac{24i\frac{\hbar}{a_0}\left(z^2-3i\frac{\hbar}{a_0}z-\frac{\hbar^2}{a_0^2}\right)z}{\left(z+i\frac{\hbar}{a_0}\right)^7}\right]\Bigg|_{z=i\hbar/a_0} \\
&= 8\pi i\left[\frac{i\frac{\hbar}{a_0}\left(-\frac{\hbar^2}{a_0^2}+3\frac{\hbar^2}{a_0^2}-\frac{\hbar^2}{a_0^2}\right)i\frac{\hbar}{a_0}}{\left(i\frac{2\hbar}{a_0}\right)^7}\right] \\
&= 8\pi i\left[\frac{-\frac{\hbar^2}{a_0^2}\left(\frac{\hbar^2}{a_0^2}\right)}{-128i\frac{\hbar^7}{a_0^7}}\right] \\
&= \frac{\pi a_0^3}{16\hbar^3} \qquad (4)
\end{aligned}$$

Parameterize the semicircular arc C_R.

$$C_R: \quad z = Re^{i\theta}, \quad \theta = 0 \quad \to \quad \theta = \pi$$

As a result, equation (4) becomes

$$\int_{-R}^{R} \frac{z^4}{\left(z^2+\frac{\hbar^2}{a_0^2}\right)^4}\,dz + \int_0^{\pi} \frac{(Re^{i\theta})^4}{\left[(Re^{i\theta})^2+\frac{\hbar^2}{a_0^2}\right]^4}(iRe^{i\theta}\,d\theta) = \frac{\pi a_0^3}{16\hbar^3}.$$

Consider the magnitude of this second integral on the left.

$$\begin{aligned}
\left|\int_0^{\pi} \frac{(Re^{i\theta})^4}{\left[(Re^{i\theta})^2+\frac{\hbar^2}{a_0^2}\right]^4}(iRe^{i\theta}\,d\theta)\right| &\le \int_0^{\pi}\left|\frac{(Re^{i\theta})^4}{\left[(Re^{i\theta})^2+\frac{\hbar^2}{a_0^2}\right]^4}(iRe^{i\theta})\right|d\theta \\
&= \int_0^{\pi} \frac{|Re^{i\theta}|^4}{\left|(Re^{i\theta})^2+\frac{\hbar^2}{a_0^2}\right|^4}|iRe^{i\theta}|\,d\theta \\
&= \int_0^{\pi} \frac{R^4}{\left|(Re^{i\theta})^2+\frac{\hbar^2}{a_0^2}\right|^4}(R)\,d\theta
\end{aligned}$$

Use the fact that for two complex numbers, z_1 and z_2, $|z_1 + z_2| \geq ||z_1| - |z_2||$.

$$\begin{aligned}\left|\int_0^\pi \frac{(Re^{i\theta})^4}{\left[(Re^{i\theta})^2 + \frac{\hbar^2}{a_0^2}\right]^4}(iRe^{i\theta}\,d\theta)\right| &\leq \int_0^\pi \frac{R^4}{\left||Re^{i\theta}|^2 - \frac{\hbar^2}{a_0^2}\right|^4}(R)\,d\theta \\ &= \int_0^\pi \frac{R^4}{\left(R^2 - \frac{\hbar^2}{a_0^2}\right)^4}(R)\,d\theta \\ &= \frac{R^4}{\left(R^2 - \frac{\hbar^2}{a_0^2}\right)^4}(\pi R)\end{aligned}$$

Notice that the upper bound of this integral's magnitude tends to zero in the limit as $R \to \infty$.

$$\lim_{R\to\infty} \frac{R^4}{\left(R^2 - \frac{\hbar^2}{a_0^2}\right)^4}(\pi R) = \lim_{R\to\infty} \frac{\pi R^5}{R^8\left(1 - \frac{\hbar^2}{a_0^2 R^2}\right)^4} = \lim_{R\to\infty} \frac{\frac{\pi}{R^3}}{\left(1 - \frac{\hbar^2}{a_0^2 R^2}\right)^4} = \frac{0}{1} = 0$$

In taking the limit of both sides of equation (4) as $R \to \infty$, then, the integral over C_R vanishes,

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{z^4}{\left(z^2 + \frac{\hbar^2}{a_0^2}\right)^4}\,dz + \underbrace{\lim_{R\to\infty}\int_{C_R} \frac{z^4}{\left(z^2 + \frac{\hbar^2}{a_0^2}\right)^4}\,dz}_{=0} = \lim_{R\to\infty} \frac{\pi a_0^3}{16\hbar^3},$$

which means

$$\int_{-\infty}^{\infty} \frac{z^4}{\left(z^2 + \frac{\hbar^2}{a_0^2}\right)^4}\,dz = \frac{\pi a_0^3}{16\hbar^3}.$$

Equation (3) then becomes

$$\begin{aligned}\langle p^2 \rangle &= \frac{16\hbar^5}{\pi a_0^5}\int_{-\infty}^{\infty} \frac{p^4}{\left(p^2 + \frac{\hbar^2}{a_0^2}\right)^4}\,dp \\ &= \frac{16\hbar^5}{\pi a_0^5}\left(\frac{\pi a_0^3}{16\hbar^3}\right) \\ &= \frac{\hbar^2}{a_0^2}.\end{aligned}$$

Part (d)

The expectation value of the kinetic energy in the ground state of hydrogen is

$$\langle T \rangle = \left\langle \frac{p^2}{2m_e} \right\rangle = \frac{1}{2m_e}\langle p^2 \rangle = \frac{1}{2m_e}\left(\frac{\hbar^2}{a_0^2}\right) = \frac{\hbar^2}{2m_e a_0^2} = \frac{\hbar^2}{2m_e}\left(\frac{m_e e^2}{4\pi\epsilon_0 \hbar^2}\right)^2 = \frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = -E_1,$$

which is consistent with the virial theorem.