Problem 4.61

Find the matrix representing S_x for a particle of spin 3/2 (using as your basis the eigenstates of S_z). Solve the characteristic equation to determine the eigenvalues of S_x.

Solution

A particle with spin 3/2 has $s = 3/2$, which means $m_s = -3/2$ or $m_s = -1/2$ or $m_s = 1/2$ or $m_s = 3/2$ (Equation 4.137 on page 166). Spin eigenstates are denoted by $|s\, m_s\rangle$; with this in mind, there are four possibilities.

$$\left|\frac{3}{2}\,\frac{3}{2}\right\rangle$$

$$\left|\frac{3}{2}\,\frac{1}{2}\right\rangle$$

$$\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle$$

$$\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle$$

If

$$\chi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ represents } \left|\frac{3}{2}\,\frac{3}{2}\right\rangle$$

$$\chi_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ represents } \left|\frac{3}{2}\,\frac{1}{2}\right\rangle$$

$$\chi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ represents } \left|\frac{3}{2}\,\frac{-1}{2}\right\rangle$$

$$\chi_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ represents } \left|\frac{3}{2}\,\frac{-3}{2}\right\rangle,$$

then the general spin state for the particle can be written as

$$\chi = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + B\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + D\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = A\chi_1 + B\chi_2 + C\chi_3 + D\chi_4,$$

where $\langle\chi|\chi\rangle = \chi^\dagger\chi = |A|^2 + |B|^2 + |C|^2 + |D|^2 = 1$ because the spinor must be normalized. Use Equation 4.135 on page 166 to determine the matrix equations involving S^2.

$$S^2|s\,m_s\rangle = \hbar^2 s(s+1)|s\,m_s\rangle \quad \rightarrow \quad \begin{cases} S^2\left|\frac{3}{2}\,\frac{3}{2}\right\rangle = \frac{15\hbar^2}{4}\left|\frac{3}{2}\,\frac{3}{2}\right\rangle \\ S^2\left|\frac{3}{2}\,\frac{1}{2}\right\rangle = \frac{15\hbar^2}{4}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle \\ S^2\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle = \frac{15\hbar^2}{4}\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle \\ S^2\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle = \frac{15\hbar^2}{4}\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}^2\chi_1 = \frac{15\hbar^2}{4}\chi_1 \\ \mathsf{S}^2\chi_2 = \frac{15\hbar^2}{4}\chi_2 \\ \mathsf{S}^2\chi_3 = \frac{15\hbar^2}{4}\chi_3 \\ \mathsf{S}^2\chi_4 = \frac{15\hbar^2}{4}\chi_4 \end{cases}$$

These four matrix equations yield a system of equations for the matrix elements of S^2.

$$\begin{cases} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{15\hbar^2}{4}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} = \frac{15\hbar^2}{4}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{11} = 15\hbar^2/4 \\ a_{21} = 0 \\ a_{31} = 0 \\ a_{41} = 0 \end{cases} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{15\hbar^2}{4}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} = \frac{15\hbar^2}{4}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{12} = 0 \\ a_{22} = 15\hbar^2/4 \\ a_{32} = 0 \\ a_{42} = 0 \end{cases} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{15\hbar^2}{4}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} = \frac{15\hbar^2}{4}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{13} = 0 \\ a_{23} = 0 \\ a_{33} = 15\hbar^2/4 \\ a_{43} = 0 \end{cases} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{15\hbar^2}{4}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \frac{15\hbar^2}{4}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{14} = 0 \\ a_{24} = 0 \\ a_{34} = 0 \\ a_{44} = 15\hbar^2/4 \end{cases} \end{cases}$$

Therefore, using $\left|\frac{3}{2}\,\frac{3}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{1}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle$ as a basis, the matrix representing S^2 for a particle of spin 3/2 is

$$\boxed{\mathsf{S}^2 = \frac{15\hbar^2}{4}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.}$$

Use Equation 4.135 on page 166 to determine the matrix equations involving S_z.

$$S_z|s\,m_s\rangle = \hbar m_s|s\,m_s\rangle \quad \to \quad \begin{cases} S_z\left|\frac{3}{2}\,\frac{3}{2}\right\rangle = \frac{3\hbar}{2}\left|\frac{3}{2}\,\frac{3}{2}\right\rangle \\ S_z\left|\frac{3}{2}\,\frac{1}{2}\right\rangle = \frac{\hbar}{2}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle \\ S_z\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle = -\frac{\hbar}{2}\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle \\ S_z\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle = -\frac{3\hbar}{2}\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_z\chi_1 = \frac{3\hbar}{2}\chi_1 \\ \mathsf{S}_z\chi_2 = \frac{\hbar}{2}\chi_2 \\ \mathsf{S}_z\chi_3 = -\frac{\hbar}{2}\chi_3 \\ \mathsf{S}_z\chi_4 = -\frac{3\hbar}{2}\chi_4 \end{cases}$$

These four matrix equations yield a system of equations for the matrix elements of S_z.

$$\begin{cases} \begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{3\hbar}{2}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \to \quad \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\\a_{41}\end{bmatrix} = \frac{3\hbar}{2}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{11} = 3\hbar/2 \\ a_{21} = 0 \\ a_{31} = 0 \\ a_{41} = 0 \end{cases} \\ \begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad \to \quad \begin{bmatrix}a_{12}\\a_{22}\\a_{32}\\a_{42}\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{12} = 0 \\ a_{22} = \hbar/2 \\ a_{32} = 0 \\ a_{42} = 0 \end{cases} \\ \begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = -\frac{\hbar}{2}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \to \quad \begin{bmatrix}a_{13}\\a_{23}\\a_{33}\\a_{43}\end{bmatrix} = -\frac{\hbar}{2}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{13} = 0 \\ a_{23} = 0 \\ a_{33} = -\hbar/2 \\ a_{43} = 0 \end{cases} \\ \begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = -\frac{3\hbar}{2}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \quad \to \quad \begin{bmatrix}a_{14}\\a_{24}\\a_{34}\\a_{44}\end{bmatrix} = -\frac{3\hbar}{2}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{14} = 0 \\ a_{24} = 0 \\ a_{34} = 0 \\ a_{44} = -3\hbar/2 \end{cases} \end{cases}$$

Therefore, using $\left|\frac{3}{2}\,\frac{3}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{1}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle$ as a basis, the matrix representing S_z for a particle of spin 3/2 is

$$\boxed{\mathsf{S}_z = \frac{\hbar}{2}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.}$$

The operators, S_x and S_y, are defined in terms of the raising and lowering operators, S_+ and S_-, by

$$\begin{cases} S_+ = S_x + iS_y \\ S_- = S_x - iS_y \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_+ = \mathsf{S}_x + i\mathsf{S}_y \\ \mathsf{S}_- = \mathsf{S}_x - i\mathsf{S}_y \end{cases}.$$

Add the respective sides of these equations to eliminate S_y.

$$\mathsf{S}_+ + \mathsf{S}_- = 2\mathsf{S}_x \quad \rightarrow \quad \mathsf{S}_x = \frac{1}{2}(\mathsf{S}_+ + \mathsf{S}_-) \tag{1}$$

Subtract the respective sides of these equations to eliminate S_x.

$$\mathsf{S}_+ - \mathsf{S}_- = 2i\mathsf{S}_y \quad \rightarrow \quad \mathsf{S}_y = \frac{1}{2i}(\mathsf{S}_+ - \mathsf{S}_-) \tag{2}$$

Use Equation 4.136 on page 166 to determine the matrix equations involving S_+.

$$S_+|s\,m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s+1)}\,|s\ (m_s{+}1)\rangle \quad \rightarrow \quad \begin{cases} S_+\left|\frac{3}{2}\,\frac{3}{2}\right\rangle = 0\left|\frac{3}{2}\,\frac{5}{2}\right\rangle \\ S_+\left|\frac{3}{2}\,\frac{1}{2}\right\rangle = \hbar\sqrt{3}\left|\frac{3}{2}\,\frac{3}{2}\right\rangle \\ S_+\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle = 2\hbar\left|\frac{3}{2}\,\frac{1}{2}\right\rangle \\ S_+\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle = \hbar\sqrt{3}\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_+\chi_1 = \mathbf{0} \\ \mathsf{S}_+\chi_2 = \hbar\sqrt{3}\,\chi_1 \\ \mathsf{S}_+\chi_3 = 2\hbar\chi_2 \\ \mathsf{S}_+\chi_4 = \hbar\sqrt{3}\,\chi_3 \end{cases}$$

These four matrix equations yield a system of equations for the matrix elements of S_+.

$$\begin{cases} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \rightarrow \quad \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{11} = 0 \\ a_{21} = 0 \\ a_{31} = 0 \\ a_{41} = 0 \end{cases} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{3} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \rightarrow \quad \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} = \hbar\sqrt{3} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{12} = \hbar\sqrt{3} \\ a_{22} = 0 \\ a_{32} = 0 \\ a_{42} = 0 \end{cases} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 2\hbar \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \rightarrow \quad \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} = 2\hbar \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{13} = 0 \\ a_{23} = 2\hbar \\ a_{33} = 0 \\ a_{43} = 0 \end{cases} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \hbar\sqrt{3} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & \rightarrow \quad \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \hbar\sqrt{3} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{14} = 0 \\ a_{24} = 0 \\ a_{34} = \hbar\sqrt{3} \\ a_{44} = 0 \end{cases} \end{cases}$$

Therefore, using $\left|\frac{3}{2}\,\frac{3}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{1}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle$ as a basis, the matrix representing S_+ for a particle of spin 3/2 is

$$\boxed{\mathsf{S}_+ = \hbar \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}.}$$

Use Equation 4.136 on page 166 to determine the matrix equations involving S_-.

$$S_-|s\,m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s-1)}\,|s\,(m_s{-}1)\rangle \quad \rightarrow \quad \begin{cases} S_-\left|\dfrac{3}{2}\,\dfrac{3}{2}\right\rangle = \hbar\sqrt{3}\left|\dfrac{3}{2}\,\dfrac{1}{2}\right\rangle \\ S_-\left|\dfrac{3}{2}\,\dfrac{1}{2}\right\rangle = 2\hbar\left|\dfrac{3}{2}\,\dfrac{-1}{2}\right\rangle \\ S_-\left|\dfrac{3}{2}\,\dfrac{-1}{2}\right\rangle = \hbar\sqrt{3}\left|\dfrac{3}{2}\,\dfrac{-3}{2}\right\rangle \\ S_-\left|\dfrac{3}{2}\,\dfrac{-3}{2}\right\rangle = 0\left|\dfrac{3}{2}\,\dfrac{-5}{2}\right\rangle \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_-\chi_1 = \hbar\sqrt{3}\,\chi_2 \\ \mathsf{S}_-\chi_2 = 2\hbar\chi_3 \\ \mathsf{S}_-\chi_3 = \hbar\sqrt{3}\,\chi_4 \\ \mathsf{S}_-\chi_4 = \mathbf{0} \end{cases}$$

These four matrix equations yield a system of equations for the matrix elements of S_-.

$$\begin{cases} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{3} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} = \hbar\sqrt{3} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \Rightarrow & \begin{cases} a_{11} = 0 \\ a_{21} = \hbar\sqrt{3} \\ a_{31} = 0 \\ a_{41} = 0 \end{cases} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2\hbar \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} = 2\hbar \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & \Rightarrow & \begin{cases} a_{12} = 0 \\ a_{22} = 0 \\ a_{32} = 2\hbar \\ a_{42} = 0 \end{cases} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \hbar\sqrt{3} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \rightarrow & \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} = \hbar\sqrt{3} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \Rightarrow & \begin{cases} a_{13} = 0 \\ a_{23} = 0 \\ a_{33} = 0 \\ a_{43} = \hbar\sqrt{3} \end{cases} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \Rightarrow & \begin{cases} a_{14} = 0 \\ a_{24} = 0 \\ a_{34} = 0 \\ a_{44} = 0 \end{cases} \end{cases}$$

Therefore, using $\left|\frac{3}{2}\,\frac{3}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{1}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle$ and $\left|\frac{3}{2}\,\frac{-3}{2}\right\rangle$ as a basis, the matrix representing S_- for a particle of spin 3/2 is

$$\boxed{\mathsf{S}_- = \hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}}.$$

By Equation (2), using the same basis, the matrix representing S_y for a particle of spin 3/2 is

$$\mathsf{S}_y = \frac{1}{2i}(\mathsf{S}_+ - \mathsf{S}_-) = \frac{1}{2i}\left(\hbar \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} - \hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\right)$$

$$= \frac{\hbar}{2i} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix}$$

$$\boxed{\mathsf{S}_y = \frac{i\hbar}{2} \begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}}.$$

And by Equation (1), using the same basis, the matrix representing S_x for a particle of spin 3/2 is

$$\mathsf{S}_x = \frac{1}{2}(\mathsf{S}_+ + \mathsf{S}_-)$$

$$= \frac{1}{2}\left(\hbar\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \hbar\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\hbar\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\right)$$

$$\boxed{\mathsf{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}.}$$

The eigenvalues of this matrix satisfy

$$\det(\mathsf{S}_x - \lambda\mathsf{I}) = 0$$

$$\det\left(\frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} -\lambda & \frac{\hbar\sqrt{3}}{2} & 0 & 0 \\ \frac{\hbar\sqrt{3}}{2} & -\lambda & \hbar & 0 \\ 0 & \hbar & -\lambda & \frac{\hbar\sqrt{3}}{2} \\ 0 & 0 & \frac{\hbar\sqrt{3}}{2} & -\lambda \end{vmatrix} = 0$$

$$-\lambda\begin{vmatrix} -\lambda & \hbar & 0 \\ \hbar & -\lambda & \frac{\hbar\sqrt{3}}{2} \\ 0 & \frac{\hbar\sqrt{3}}{2} & -\lambda \end{vmatrix} - \frac{\hbar\sqrt{3}}{2}\begin{vmatrix} \frac{\hbar\sqrt{3}}{2} & \hbar & 0 \\ 0 & -\lambda & \frac{\hbar\sqrt{3}}{2} \\ 0 & \frac{\hbar\sqrt{3}}{2} & -\lambda \end{vmatrix} = 0$$

$$-\lambda\left(-\lambda\begin{vmatrix} -\lambda & \frac{\hbar\sqrt{3}}{2} \\ \frac{\hbar\sqrt{3}}{2} & -\lambda \end{vmatrix} - \hbar\begin{vmatrix} \hbar & \frac{\hbar\sqrt{3}}{2} \\ 0 & -\lambda \end{vmatrix}\right) - \frac{\hbar\sqrt{3}}{2}\left(\frac{\hbar\sqrt{3}}{2}\begin{vmatrix} -\lambda & \frac{\hbar\sqrt{3}}{2} \\ \frac{\hbar\sqrt{3}}{2} & -\lambda \end{vmatrix}\right) = 0.$$

Solve the equation for λ.

$$\lambda\left[\lambda\left(\lambda^2-\frac{3\hbar^2}{4}\right)+\hbar(-\hbar\lambda)\right]-\frac{3\hbar^2}{4}\left(\lambda^2-\frac{3\hbar^2}{4}\right)=0$$

$$\lambda^4-\frac{5\hbar^2}{2}\lambda^2+\frac{9\hbar^4}{16}=0$$

$$\left(\lambda+\frac{3\hbar}{2}\right)\left(\lambda+\frac{\hbar}{2}\right)\left(\lambda-\frac{\hbar}{2}\right)\left(\lambda-\frac{3\hbar}{2}\right)=0$$

$$\boxed{\lambda=\left\{-\frac{3\hbar}{2},-\frac{\hbar}{2},\frac{\hbar}{2},\frac{3\hbar}{2}\right\}}$$

These are the eigenvalues of S_x—the possible values one would obtain if he or she measured S_x on a particle of spin 3/2. The eigenvalues of S_y and S_z happen to be the same.

$$\det(\mathsf{S}_y-\lambda_y\mathsf{I})=0$$

$$\begin{vmatrix} -\lambda_y & -\frac{i\hbar\sqrt{3}}{2} & 0 & 0 \\ \frac{i\hbar\sqrt{3}}{2} & -\lambda_y & -i\hbar & 0 \\ 0 & i\hbar & -\lambda_y & -\frac{i\hbar\sqrt{3}}{2} \\ 0 & 0 & \frac{i\hbar\sqrt{3}}{2} & -\lambda_y \end{vmatrix}=0$$

$$\lambda_y^4-\frac{5\hbar^2}{2}\lambda_y^2+\frac{9\hbar^4}{16}=0$$

$$\boxed{\lambda_y=\left\{-\frac{3\hbar}{2},-\frac{\hbar}{2},\frac{\hbar}{2},\frac{3\hbar}{2}\right\}}$$

$$\det(\mathsf{S}_z-\lambda_z\mathsf{I})=0$$

$$\begin{vmatrix} \frac{3\hbar}{2}-\lambda_z & 0 & 0 & 0 \\ 0 & \frac{\hbar}{2}-\lambda_z & 0 & 0 \\ 0 & 0 & -\frac{\hbar}{2}-\lambda_z & 0 \\ 0 & 0 & 0 & -\frac{3\hbar}{2}-\lambda_z \end{vmatrix}=0$$

$$\left(\frac{3\hbar}{2}-\lambda_z\right)\left(\frac{\hbar}{2}-\lambda_z\right)\left(-\frac{\hbar}{2}-\lambda_z\right)\left(-\frac{3\hbar}{2}-\lambda_z\right)=0$$

$$\boxed{\lambda_z=\left\{-\frac{3\hbar}{2},-\frac{\hbar}{2},\frac{\hbar}{2},\frac{3\hbar}{2}\right\}}$$