Problem 4.64

The electron in a hydrogen atom occupies the combined spin and position state

$$R_{21}\left(\sqrt{1/3}\,Y_1^0\chi_+ + \sqrt{2/3}\,Y_1^1\chi_-\right).$$

(a) If you measured the orbital angular momentum squared (L^2), what values might you get, and what is the probability of each?

(b) Same for the z component of orbital angular momentum (L_z).

(c) Same for the spin angular momentum squared (S^2).

(d) Same for the z component of spin angular momentum (S_z).

Let $\mathbf{J} \equiv \mathbf{L} + \mathbf{S}$ be the *total* angular momentum.

(e) If you measured J^2, what values might you get, and what is the probability of each?

(f) Same for J_z.

(g) If you measured the *position* of the particle, what is the probability density for finding it at r, θ, ϕ?

(h) If you measured both the z component of the spin *and* the distance from the origin (note that these are compatible observables), what is the probability per unit r for finding the particle with spin up and at radius r?

Solution

The state of the electron in a hydrogen atom is given.

$$\mathbf{\Psi}_e(r,\theta,\phi) = R_{21}(r)\left[\frac{1}{\sqrt{3}}Y_1^0(\theta,\phi)\chi_+ + \sqrt{\frac{2}{3}}\,Y_1^1(\theta,\phi)\chi_-\right]$$

Part (a)

Apply L^2 to each of the eigenstates to determine the possible measurements of the orbital angular momentum squared of the electron.

$$L^2Y_\ell^{m_\ell} = \hbar^2\ell(\ell+1)Y_\ell^{m_\ell} \quad \rightarrow \quad \begin{cases} L^2Y_1^0 = 1(1+1)\hbar^2Y_1^0 \\ \\ L^2Y_1^1 = 1(1+1)\hbar^2Y_1^1 \end{cases}$$

Therefore, the possible measurements and their corresponding probabilities are

$$2\hbar^2 \quad \text{with probability} \quad \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

$$2\hbar^2 \quad \text{with probability} \quad \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3},$$

that is, a measurement of $2\hbar^2$ with probability 1.

Part (b)

Apply L_z to each of the eigenstates to determine the possible measurements of the z-component of orbital angular momentum of the electron.

$$L_z Y_\ell^{m_\ell} = \hbar m_\ell Y_\ell^{m_\ell} \quad \rightarrow \quad \begin{cases} L_z Y_1^0 = \hbar(0) Y_1^0 \\ \\ L_z Y_1^1 = \hbar(1) Y_1^1 \end{cases}$$

Therefore, the possible measurements and their corresponding probabilities are

$$0 \quad \text{with probability} \quad \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

$$\hbar \quad \text{with probability} \quad \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}.$$

Part (c)

Rewrite the wave function in terms of the eigenstates of S^2.

$$\mathbf{\Psi}_e(r,\theta,\phi) = R_{21}(r)\left[\frac{1}{\sqrt{3}} Y_1^0(\theta,\phi)\left|\frac{1}{2}\,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\, Y_1^1(\theta,\phi)\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right]$$

Apply S^2 to each of the eigenstates to determine the possible measurements of the spin angular momentum squared of the electron.

$$S^2|s\ m_s\rangle = \hbar^2 s(s+1)|s\ m_s\rangle \quad \rightarrow \quad \begin{cases} S^2\left|\frac{1}{2}\,\frac{1}{2}\right\rangle = \frac{1}{2}\left(\frac{3}{2}\right)\hbar^2\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\ \\ S^2\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle = \frac{1}{2}\left(\frac{3}{2}\right)\hbar^2\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \end{cases}$$

Therefore, the possible measurements and their corresponding probabilities are

$$\frac{3\hbar^2}{4} \quad \text{with probability} \quad \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

$$\frac{3\hbar^2}{4} \quad \text{with probability} \quad \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3},$$

that is, a measurement of $3\hbar^2/4$ with probability 1.

Part (d)

Rewrite the wave function in terms of the eigenstates of S_z.

$$\mathbf{\Psi}_e(r,\theta,\phi) = R_{21}(r)\left[\frac{1}{\sqrt{3}}Y_1^0(\theta,\phi)\left|\frac{1}{2}\,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\,Y_1^1(\theta,\phi)\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right]$$

Apply S_z to each of the eigenstates to determine the possible measurements of the z-component of spin angular momentum of the electron.

$$S_z|s\ m_s\rangle = \hbar m_s|s\ m_s\rangle \quad \rightarrow \quad \begin{cases} S_z\left|\frac{1}{2}\,\frac{1}{2}\right\rangle = \hbar\left(\frac{1}{2}\right)\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\ S_z\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle = \hbar\left(\frac{-1}{2}\right)\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle \end{cases}$$

Therefore, the possible measurements and their corresponding probabilities are

$$\frac{\hbar}{2} \quad \text{with probability} \quad \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

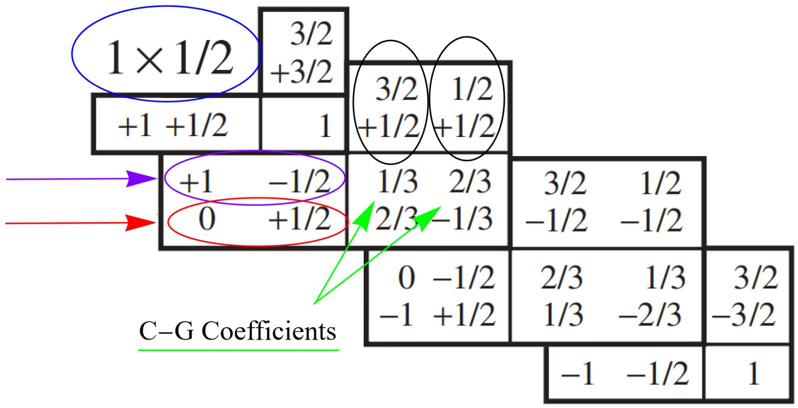
$$-\frac{\hbar}{2} \quad \text{with probability} \quad \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}.$$

Part (e)

Rewrite the wave function in terms of the eigenstates of J^2 by using the appropriate Clebsch–Gordon table on page 179.

$$\begin{aligned}
\mathbf{\Psi}_e(r,\theta,\phi) &= R_{21}(r)\left[\frac{1}{\sqrt{3}}Y_1^0(\theta,\phi)\chi_+ + \sqrt{\frac{2}{3}}\,Y_1^1(\theta,\phi)\chi_-\right] \\
&= R_{21}(r)\left(\frac{1}{\sqrt{3}}|1\,0\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\,|1\,1\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right) \\
&= R_{21}(r)\left(\frac{1}{\sqrt{3}}\left|1\,\frac{1}{2}\,0\,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|1\,\frac{1}{2}\,1\,\frac{-1}{2}\right\rangle\right) \\
&= R_{21}(r)\left[\frac{1}{\sqrt{3}}\left(\sqrt{\frac{2}{3}}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right) + \sqrt{\frac{2}{3}}\left(\sqrt{\frac{1}{3}}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right)\right] \\
&= R_{21}(r)\left(\frac{2\sqrt{2}}{3}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle + \frac{1}{3}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right)
\end{aligned}$$

$$\left|1\ \frac{1}{2}\ 1\ \frac{-1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle$$



$$\left|1\ \frac{1}{2}\ 0\ \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle$$

Apply J^2 to each of the eigenstates to determine the possible measurements of the total angular momentum squared of the electron.

$$J^2|j\ m\rangle = \hbar^2 j(j+1)|j\ m\rangle \quad \rightarrow \quad \begin{cases} J^2\left|\frac{3}{2}\,\frac{1}{2}\right\rangle = \frac{3}{2}\left(\frac{5}{2}\right)\hbar^2\left|\frac{3}{2}\,\frac{1}{2}\right\rangle \\ \\ J^2\left|\frac{1}{2}\,\frac{1}{2}\right\rangle = \frac{1}{2}\left(\frac{3}{2}\right)\hbar^2\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \end{cases}$$

Therefore, the possible measurements and their corresponding probabilities are

$$\frac{15\hbar^2}{4} \quad \text{with probability} \quad \left|\frac{2\sqrt{2}}{3}\right|^2 = \frac{8}{9}$$

$$\frac{3\hbar^2}{4} \quad \text{with probability} \quad \left|\frac{1}{3}\right|^2 = \frac{1}{9}.$$

Part (f)

The eigenstates of J^2 are also the eigenstates of J_z, so use the wave function in part (e) and apply J_z to each of the eigenstates to determine the possible measurements of the z-component of the total angular momentum of the electron.

$$J_z|j\ m\rangle = \hbar m|j\ m\rangle \quad \rightarrow \quad \begin{cases} J_z\left|\frac{3}{2}\,\frac{1}{2}\right\rangle = \hbar\left(\frac{1}{2}\right)\left|\frac{3}{2}\,\frac{1}{2}\right\rangle \\ \\ J_z\left|\frac{1}{2}\,\frac{1}{2}\right\rangle = \hbar\left(\frac{1}{2}\right)\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \end{cases}$$

Therefore, the possible measurements and their corresponding probabilities are

$$\frac{\hbar}{2} \quad \text{with probability} \quad \left|\frac{2\sqrt{2}}{3}\right|^2 = \frac{8}{9}$$

$$\frac{\hbar}{2} \quad \text{with probability} \quad \left|\frac{1}{3}\right|^2 = \frac{1}{9},$$

that is, a measurement of $\hbar/2$ with probability 1.

Part (g)

Write the given state of the electron in matrix form.

$$\begin{aligned}
\mathbf{\Psi}_e(r,\theta,\phi) &= R_{21}(r)\left[\frac{1}{\sqrt{3}}Y_1^0(\theta,\phi)\chi_+ + \sqrt{\frac{2}{3}}\,Y_1^1(\theta,\phi)\chi_-\right] \\
&= \frac{1}{\sqrt{3}}R_{21}(r)Y_1^0(\theta,\phi)\begin{bmatrix}1\\0\end{bmatrix} + \sqrt{\frac{2}{3}}\,R_{21}(r)Y_1^1(\theta,\phi)\begin{bmatrix}0\\1\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{\sqrt{3}}R_{21}(r)Y_1^0(\theta,\phi) \\ \sqrt{\frac{2}{3}}\,R_{21}(r)Y_1^1(\theta,\phi)\end{bmatrix}
\end{aligned}$$

If the position of the electron is measured, the probability density for finding it at r, θ, ϕ is

$$\begin{aligned}
\mathbf{\Psi}_e^\dagger\mathbf{\Psi}_e &= \begin{bmatrix}\frac{1}{\sqrt{3}}R_{21}^*(r)Y_1^0(\theta,\phi)^* & \sqrt{\frac{2}{3}}\,R_{21}^*(r)Y_1^1(\theta,\phi)^*\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{3}}R_{21}(r)Y_1^0(\theta,\phi) \\ \sqrt{\frac{2}{3}}\,R_{21}(r)Y_1^1(\theta,\phi)\end{bmatrix} \\
&= \frac{1}{3}\left|R_{21}(r)\right|^2\left|Y_1^0(\theta,\phi)\right|^2 + \frac{2}{3}\left|R_{21}(r)\right|^2\left|Y_1^1(\theta,\phi)\right|^2 \\
&= \frac{1}{3}\left|R_{21}(r)\right|^2\left[\left|Y_1^0(\theta,\phi)\right|^2 + 2\left|Y_1^1(\theta,\phi)\right|^2\right] \\
&= \frac{1}{3}\left|\frac{1}{2\sqrt{6}}a_0^{-3/2}\left(\frac{r}{a_0}\right)\exp\left(-\frac{r}{2a_0}\right)\right|^2\left(\left|\sqrt{\frac{3}{4\pi}}\cos\theta\right|^2 + 2\left|-\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{i\phi}\right|^2\right) \\
&= \frac{1}{3}\left(\frac{r^2}{24a_0^5}e^{-r/a_0}\right)\left(\frac{3}{4\pi}\cos^2\theta + \frac{6}{8\pi}\sin^2\theta\right) \\
&= \frac{r^2}{96\pi a_0^5}e^{-r/a_0}.
\end{aligned}$$

Note that a_0 is the Bohr radius, and the spherical harmonics and the radial wave functions can be found on pages 137 and 151, respectively.

Part (h)

The fact that the z-component of the spin angular momentum and the distance from the origin are compatible observables means that the measurement of one does not affect the measurement of the other. Consequently, the probability of finding the electron with spin up (event A) and finding the electron at radius r (event B) is the product of the probabilities of these two events individually.

$$P(\text{A and B}) = P(A)P(B)$$

Finding the electron with spin up means measuring a value of $\hbar/2$ for the z-component of spin angular momentum, and this was found in part (d) to have a probability of 1/3.

$$P(\text{A and B}) = \frac{1}{3}P(B)$$

Now integrate the probability density obtained in part (g) over a spherical shell with radius r to get the probability per unit r of finding the electron at radius r.

$$\begin{aligned}
P(\text{A and B}) &= \frac{1}{3} \iint\limits_{\substack{\text{spherical shell} \\ \text{with radius } r}} \boldsymbol{\Psi}_e^\dagger \boldsymbol{\Psi}_e \, d\mathcal{S} \\
&= \frac{1}{3} \int_0^\pi \int_0^{2\pi} \frac{r^2}{96\pi a_0^5} e^{-r/a_0} \, (r^2 \sin\theta \, d\phi \, d\theta) \\
&= \frac{r^4}{288\pi a_0^5} e^{-r/a_0} \left(\int_0^\pi \sin\theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right) \\
&= \frac{r^4}{288\pi a_0^5} e^{-r/a_0} (2)(2\pi) \\
&= \frac{r^4}{72 a_0^5} e^{-r/a_0}
\end{aligned}$$