Problem 4.66

Deduce the condition for minimum uncertainty in S_x and S_y (that is, equality in the expression $\sigma_{S_x}\sigma_{S_y} \geq (\hbar/2)|\langle S_z\rangle|$), for a particle of spin 1/2 in the generic state (Equation 4.139). *Answer:* With no loss of generality we can pick a to be real; then the condition for minimum uncertainty is that b is either pure real or else pure imaginary.

Solution

Work backwards: Assume equality in the given inequality and determine the condition that has to be true as a result.

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}|\langle S_z\rangle|$$

$$\sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2}\sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \frac{\hbar}{2}|\langle S_z\rangle| \tag{1}$$

For the generic state in Equation 4.139 on page 167, the quantities in this equation were all calculated in Problem 4.31. The results are summarized here.

$$\langle S_x\rangle = \hbar\,\mathrm{Re}\,(a^*b) = \hbar\,\mathrm{Re}\left[\left(|a|e^{i\phi_a}\right)^*\left(|b|e^{i\phi_b}\right)\right] = \hbar\,\mathrm{Re}\left[|a||b|e^{-i(\phi_a-\phi_b)}\right] = \hbar|a||b|\cos(\phi_a-\phi_b)$$

$$\langle S_x^2\rangle = \frac{\hbar^2}{4}$$

$$\langle S_y\rangle = \hbar\,\mathrm{Im}\,(a^*b) = \hbar\,\mathrm{Im}\left[\left(|a|e^{i\phi_a}\right)^*\left(|b|e^{i\phi_b}\right)\right] = \hbar\,\mathrm{Im}\left[|a||b|e^{-i(\phi_a-\phi_b)}\right] = -\hbar|a||b|\sin(\phi_a-\phi_b)$$

$$\langle S_y^2\rangle = \frac{\hbar^2}{4}$$

$$\langle S_z\rangle = \frac{\hbar}{2}(|a|^2-|b|^2)$$

$$\langle S_z^2\rangle = \frac{\hbar^2}{4}$$

Plug them into equation (1) and simplify both sides.

$$\sqrt{\frac{\hbar^2}{4} - \hbar^2|a|^2|b|^2\cos^2(\phi_a-\phi_b)}\sqrt{\frac{\hbar^2}{4} - \hbar^2|a|^2|b|^2\sin^2(\phi_a-\phi_b)} = \frac{\hbar}{2}\left|\frac{\hbar}{2}(|a|^2-|b|^2)\right|$$

$$\frac{\hbar^2}{4}\sqrt{1-4|a|^2|b|^2\cos^2(\phi_a-\phi_b)}\sqrt{1-4|a|^2|b|^2\sin^2(\phi_a-\phi_b)} = \frac{\hbar^2}{4}\left||a|^2-|b|^2\right|$$

Multiply both sides by $4/\hbar^2$.

$$\sqrt{1-4|a|^2|b|^2\cos^2(\phi_a-\phi_b)}\sqrt{1-4|a|^2|b|^2\sin^2(\phi_a-\phi_b)} = \left||a|^2-|b|^2\right|$$

Square both sides.

$$\left[1-4|a|^2|b|^2\cos^2(\phi_a-\phi_b)\right]\left[1-4|a|^2|b|^2\sin^2(\phi_a-\phi_b)\right]=\left(|a|^2-|b|^2\right)^2$$

Simplify the left side.

$$1-4|a|^2|b|^2\overbrace{\left[\cos^2(\phi_a-\phi_b)+\sin^2(\phi_a-\phi_b)\right]}^{=1}+16|a|^4|b|^4\cos^2(\phi_a-\phi_b)\sin^2(\phi_a-\phi_b)=\left(|a|^2-|b|^2\right)^2$$

$$1-4|a|^2|b|^2+4|a|^4|b|^4\left[2\sin(\phi_a-\phi_b)\cos(\phi_a-\phi_b)\right]^2=\left(|a|^2-|b|^2\right)^2$$

$$1-4|a|^2|b|^2+4|a|^4|b|^4\sin^2[2(\phi_a-\phi_b)]=\left(|a|^2-|b|^2\right)^2$$

$$=|a|^4-2|a|^2|b|^2+|b|^4$$

Add $4|a|^2|b|^2$ to both sides. Note that the generic state is normalized, so $|a|^2+|b|^2=1$.

$$\begin{aligned}1+4|a|^4|b|^4\sin^2[2(\phi_a-\phi_b)]&=|a|^4+2|a|^2|b|^2+|b|^4\\&=\left(|a|^2+|b|^2\right)^2\\&=(1)^2\\&=1\end{aligned}$$

Subtract 1 from both sides and solve for the phase difference.

$$\begin{aligned}4|a|^4|b|^4\sin^2[2(\phi_a-\phi_b)]&=0\\ \sin[2(\phi_a-\phi_b)]&=0\\ 2(\phi_a-\phi_b)&=n\pi,\quad n=0,\pm1,\pm2,\ldots\\ \phi_a-\phi_b&=\frac{n\pi}{2}\end{aligned}$$

Therefore, the condition for minimum uncertainty in S_x and S_y for a particle of spin 1/2 in the generic state,

$$\chi=\begin{bmatrix}a\\b\end{bmatrix},$$

is that the phases of a and b differ by an integer or half-integer multiple of π. If a is real, for example, then b must either be real as well or be purely imaginary for the condition to be satisfied.