Problem 4.69

Find a few of the Bohr energies for hydrogen by "wagging the dog" (Problem 2.55), starting with Equation 4.53—or, better yet, Equation 4.56; in fact, why not use Equation 4.68 to set $\rho_0 = 2n$, and tweak n? We know that the correct solutions occur when n is a positive integer, so you might start with $n = 0.9$, 1.9, 2.9, etc., and increase it in small increments—the tail should wag when you pass 1, 2, 3, ... Find the lowest three ns, to four significant digits, first for $\ell = 0$, and then for $\ell = 1$ and $\ell = 2$. *Warning:* Mathematica doesn't like to divide by zero, so you might change ρ to $(\rho + 0.000001)$ in the denominator. *Note:* $u(0) = 0$ in all cases, but $u'(0) = 0$ only for $\ell \geq 1$ (Equation 4.59). So for $\ell = 0$ you can use $u(0) = 0$, $u'(0) = 1$. For $\ell > 0$ you might be tempted to use $u(0) = 0$ and $u'(0) = 0$, but Mathematica is lazy, and will go for the trivial solution $u(\rho) \equiv 0$; better, therefore, to use (say) $u(1) = 1$ and $u'(0) = 0$.

Solution

Equation 4.56 is on page 144. Write it with $\rho_0 = 2n$.

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{2n}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u \tag{4.56}$$

The lowest three ns will be found for $\ell = 0$, $\ell = 1$, and $\ell = 2$ by wagging the dog.

<u>$\boldsymbol{\ell = 0}$</u>

Set $\ell = 0$ in the equation and use the suggested boundary conditions.

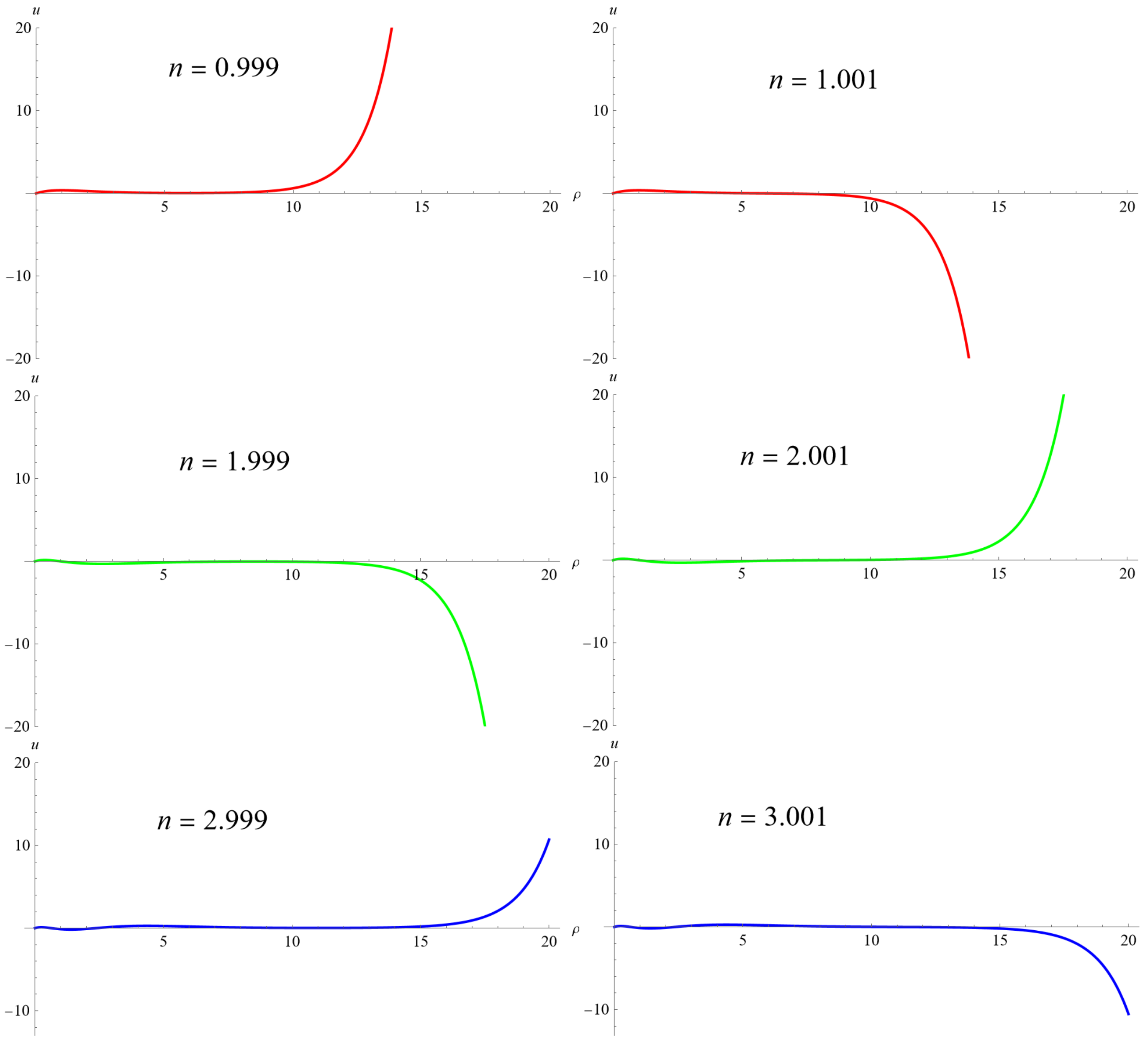
$$\frac{d^2u}{d\rho^2} = \left(1 - \frac{2n}{\rho}\right)u, \quad u(0) = 0, \ u'(0) = 1$$

Modify the code used in Problem 2.55 and enter it into Mathematica.

```
Manipulate[
  Plot[
    Evaluate[
      u[x]/.
        NDSolve[
          {u''[x] - (1 - 2n/(x+0.000001))*u[x] == 0, u[0] == 0, u'[0] == 1}, u[x], {x, 0, 20}
        ]
      ],
    {x, 0, 20}, PlotRange -> {-20, 20}
  ], {n, 0, 3.5}
]
```

This code yields a graph of the solution to the differential equation with a slider at the top. Moving the slider to the right varies n, and one can quickly see the significant values.

The important values of n are the ones before and after the tail wags.



Therefore, the lowest three values of n are 1.000, 2.000, and 3.000 for $\ell = 0$.

$\ell = 1$

Set $\ell = 1$ in the equation and use the suggested boundary conditions.

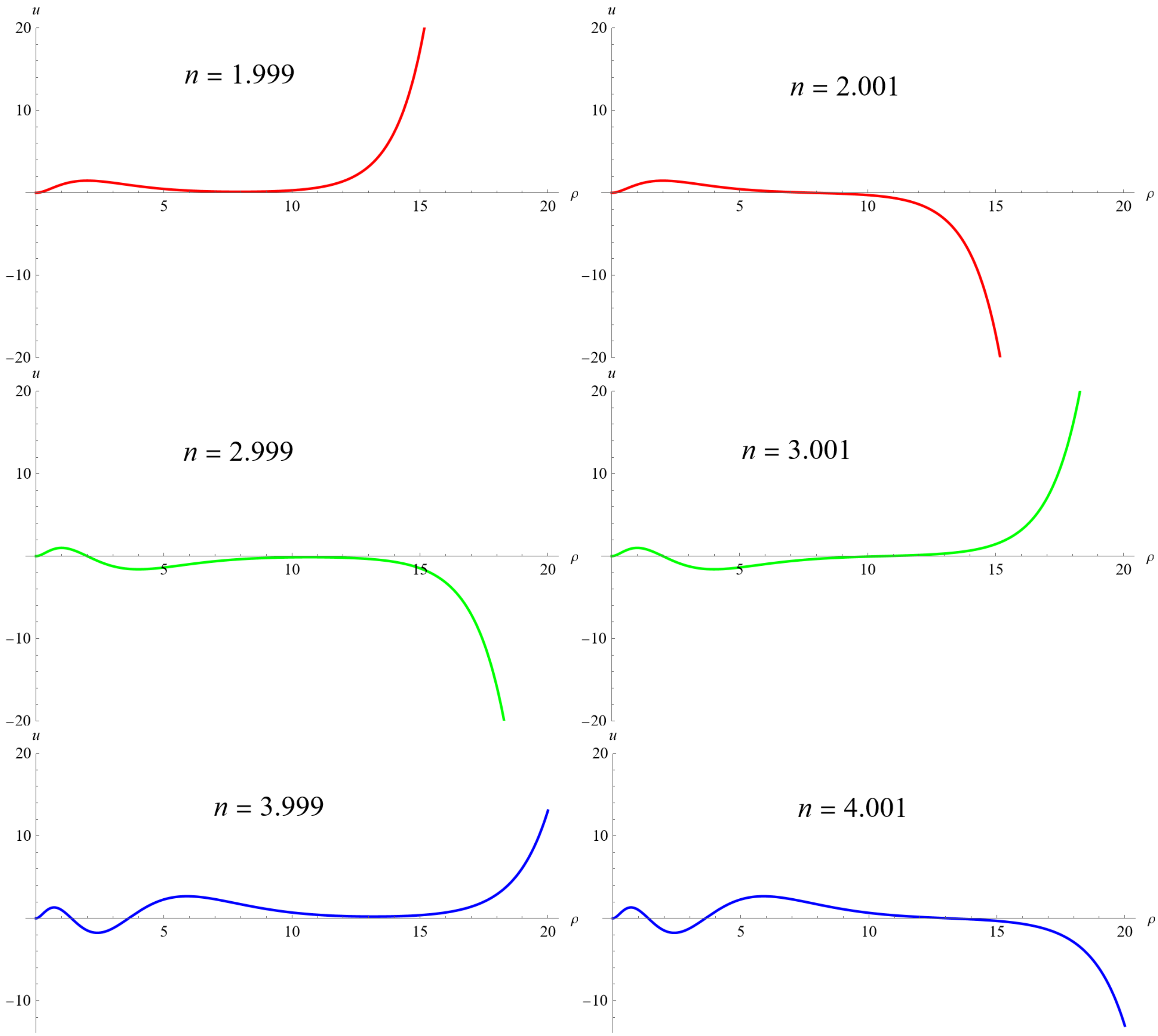
$$\frac{d^2u}{d\rho^2} = \left(1 - \frac{2n}{\rho} + \frac{2}{\rho^2}\right)u, \quad u(1) = 1,\ u'(0) = 0$$

Modify the code used in Problem 2.55 and enter it into Mathematica.

```
Manipulate[
  Plot[
    Evaluate[
      u[x]/.
        NDSolve[
          {u''[x] - (1 - 2n/(x+0.000001) + 2/(x+0.000001)^2)*u[x] == 0, u[1] == 1, u'[0] == 0}, u[x], {x, 0, 20}
        ]
      ],
    {x, 0, 20}, PlotRange -> {-20, 20}
  ], {n, 0, 5}
]
```

This code yields a graph of the solution to the differential equation with a slider at the top. Moving the slider to the right varies n, and one can quickly see the significant values.

The important values of n are the ones before and after the tail wags.



Therefore, the lowest three values of n are 2.000, 3.000, and 4.000 for $\ell = 1$.

$\ell = 2$

Set $\ell = 2$ in the equation and use the suggested boundary conditions.

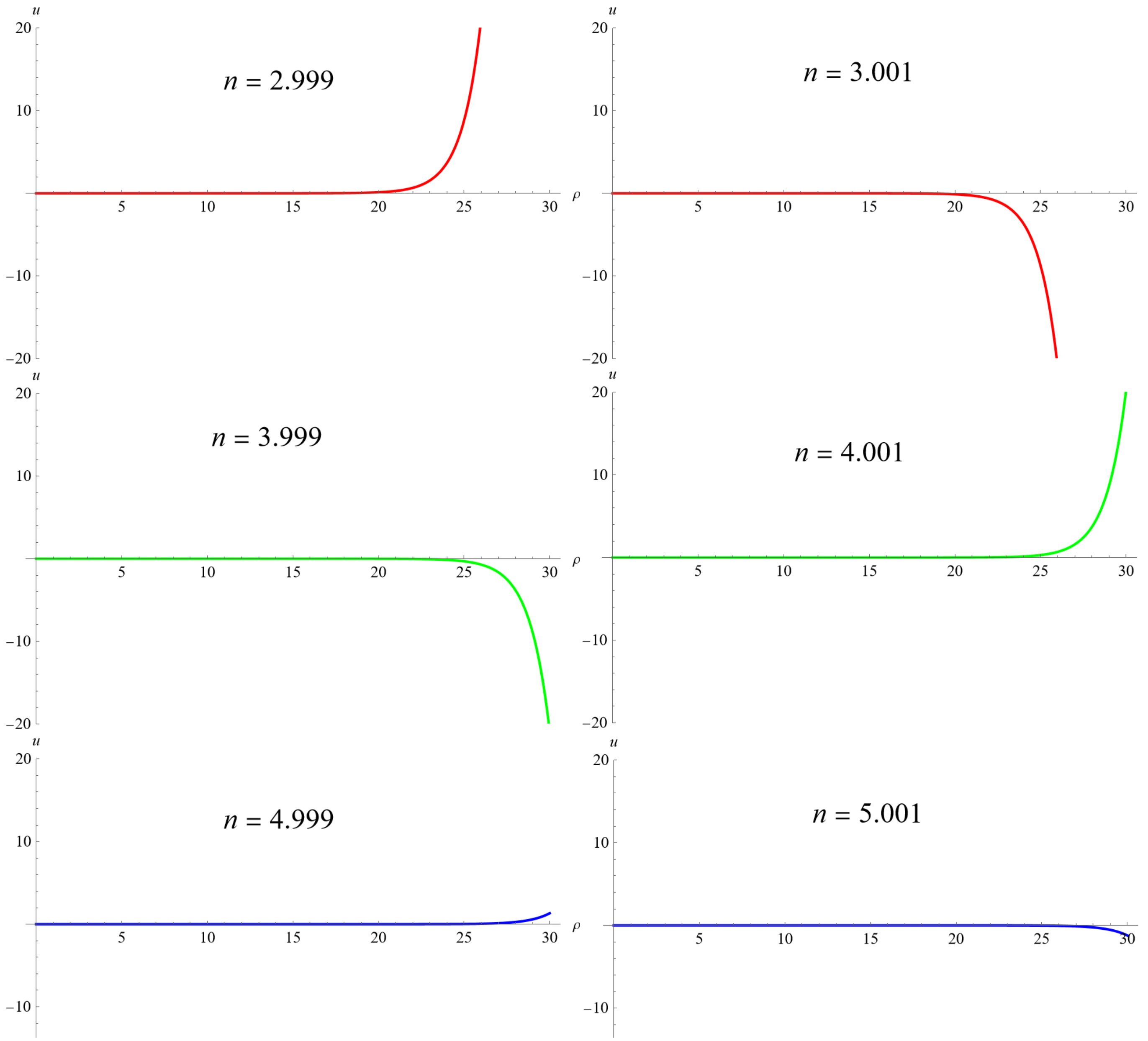
$$\frac{d^2u}{d\rho^2} = \left(1 - \frac{2n}{\rho} + \frac{6}{\rho^2}\right)u, \quad u(1) = 1, \ u'(0) = 0$$

Modify the code used in Problem 2.55 and enter it into Mathematica.

```
Manipulate[
  Plot[
    Evaluate[
      u[x]/.
        NDSolve[
          {u''[x] - (1 - 2n/(x+0.00000001) + 6/(x+0.00000001)^2)*u[x] == 0, u[1] == 1, u'[0] == 0}, u[x], {x, 0, 30}
        ]
      ],
    {x, 0, 30}, PlotRange -> {-20, 20}
  ], {n, 0, 6}
]
```

This code yields a graph of the solution to the differential equation with a slider at the top. Moving the slider to the right varies n, and one can quickly see the significant values.

The important values of n are the ones before and after the tail wags.



Therefore, the lowest three values of n are 3.000, 4.000, and 5.000 for $\ell = 2$.