Problem 4.70

Sequential Spin Measurements.

(a) At time $t = 0$ a large ensemble of spin-1/2 particles is prepared, all of them in the spin-up state (with respect to the z axis).[74] They are not subject to any forces or torques. At time $t_1 > 0$ each spin is measured—some along the z direction and others along the x direction (but we aren't told the results). At time $t_2 > t_1$ their spin is measured again, this time along the x direction, and those with spin up (along x) are saved as a subensemble (those with spin down are discarded). *Question:* Of those remaining (the subensemble), what fraction had spin up (along z or x, depending on which was measured) in the first measurement?

(b) Part (a) was easy—trivial, really, once you see it. Here's a more pithy generalization: At time $t = 0$ an ensemble of spin-1/2 particles is prepared, all in the spin-up state along direction $\boldsymbol{a}$. At time $t_1 > 0$ their spins are measured along direction $\boldsymbol{b}$ (but we are not told the results), and at time $t_2 > t_1$ their spins are measured along direction $\boldsymbol{c}$. Those with spin up (along $\boldsymbol{c}$) are saved as a subensemble. Of the particles in this subensemble, what fraction had spin up (along $\boldsymbol{b}$) in the first measurement? *Hint:* Use Equation 4.155 to show that the probability of getting spin up (along $\boldsymbol{b}$) in the first measurement is $P_+ = \cos^2(\theta_{ab}/2)$, and (by extension) the probability of getting spin up in both measurements is $P_{++} = \cos^2(\theta_{ab}/2)\cos^2(\theta_{bc}/2)$. Find the other three probabilities (P_{+-}, P_{-+}, and P_{--}). *Beware:* If the outcome of the first measurement was spin down, the relevant angle is now the supplement of θ_{bc}. *Answer:* $\left[1 + \tan^2(\theta_{ab}/2)\tan^2(\theta_{bc}/2)\right]^{-1}$.

Solution

Part (a)

A particle with spin 1/2 has $s = 1/2$, which means $m_s = -1/2$ or $m_s = 1/2$. The spin eigenstates are denoted by $|s\ m_s\rangle$, so there are two possibilities: $\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle$ and $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$. If we let

$$\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{represent} \qquad \left|\frac{1}{2}\ \frac{1}{2}\right\rangle \qquad \text{(spin up)}$$

and

$$\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{represent} \qquad \left|\frac{1}{2}\ \frac{-1}{2}\right\rangle \qquad \text{(spin down)} \qquad ,$$

then the general spin state of the spin-1/2 particle can be written as

$$\chi = \begin{bmatrix} A \\ B \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} + B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\chi_+ + B\chi_-,$$

where $\langle \chi \,|\, \chi \rangle = \chi^\dagger \chi = |A|^2 + |B|^2 = 1$ because it has to be normalized for physical relevance.

[74] N. D. Mermin, *Physics Today*, October 2011, page 8.

Since the spin angular momentum of the particle will be measured along the x- and z-directions, determine the eigenvalues and normalized eigenvectors for the matrices representing S_x and S_z with respect to the basis, $\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle$ and $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$.

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_- = -\dfrac{\hbar}{2} \\ \chi_-^{(x)} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases} \quad \text{and} \quad \begin{cases} \lambda_+ = \dfrac{\hbar}{2} \\ \chi_+^{(x)} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$$

$$\mathsf{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \lambda_- = -\dfrac{\hbar}{2} \\ \chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \quad \text{and} \quad \begin{cases} \lambda_+ = \dfrac{\hbar}{2} \\ \chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$$

Notice that the general spin state is already written in terms of the eigenvectors of S_z.

$$\chi = \begin{bmatrix} A \\ B \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} + B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\chi_+ + B\chi_-,$$

As a result, measuring S_z on a spin-1/2 particle in the general state $\chi = \begin{bmatrix} A \\ B \end{bmatrix}$ yields values of

$$\frac{\hbar}{2} \quad \text{with probability} \quad |A|^2$$

$$-\frac{\hbar}{2} \quad \text{with probability} \quad |B|^2.$$

Now expand the general spin state in terms of the eigenvectors of S_x.

$$\chi = C_1\chi_-^{(x)} + C_2\chi_+^{(x)}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{C_1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{C_2}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}} \\ -\frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}} \end{bmatrix}$$

This implies a system of two equations for the two unknowns, C_1 and C_2.

$$\begin{cases} A = \dfrac{C_1}{\sqrt{2}} + \dfrac{C_2}{\sqrt{2}} \\ B = -\dfrac{C_1}{\sqrt{2}} + \dfrac{C_2}{\sqrt{2}} \end{cases}$$

Solving it yields

$$C_1 = \frac{1}{\sqrt{2}}(A - B) \quad \text{and} \quad C_2 = \frac{1}{\sqrt{2}}(A + B).$$

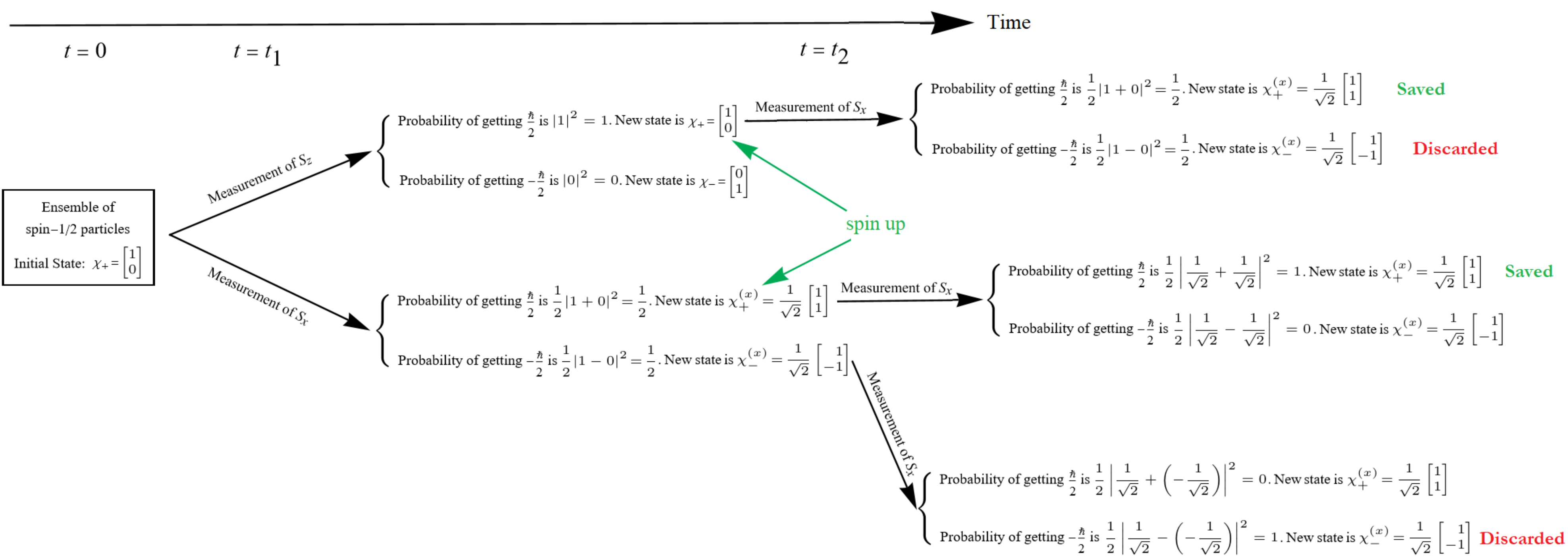
As a result, measuring S_x on a spin-1/2 particle in the general state $\chi = \begin{bmatrix} A \\ B \end{bmatrix}$ yields values of

$$\frac{\hbar}{2} \quad \text{with probability} \quad |C_2|^2 = \frac{1}{2}|A + B|^2$$

$$-\frac{\hbar}{2} \quad \text{with probability} \quad |C_1|^2 = \frac{1}{2}|A - B|^2.$$

Now consider the sequential spin measurements.

Timeline of Events



Notice that all the particles saved in the subensemble at the end came from a spin-up state previously. Therefore, the fraction of particles that had spin up in the first measurement is 1.

Part (b)

In Problem 4.33 the matrix representing the component of spin angular momentum along an arbitrary direction $\hat{r}$ was found.

$$\mathsf{S}_r = \begin{bmatrix} \frac{\hbar}{2}\cos\theta & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & -\frac{\hbar}{2}\cos\theta \end{bmatrix}$$

The eigenvalues and corresponding normalized eigenvectors were also determined.

$$\begin{cases} \lambda_- = -\dfrac{\hbar}{2} \\ \chi_-^{(r)} = \begin{bmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix} \end{cases} \quad \text{and} \quad \begin{cases} \lambda_+ = +\dfrac{\hbar}{2} \\ \chi_+^{(r)} = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} \end{cases}$$

Expand the general spin state in terms of the eigenvectors of S_r.

$$\chi = C_3\chi_-^{(r)} + C_4\chi_+^{(r)}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = C_3 \begin{bmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix} + C_4 \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -C_3e^{-i\phi}\sin\frac{\theta}{2} + C_4\cos\frac{\theta}{2} \\ C_3\cos\frac{\theta}{2} + C_4e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

This implies a system of two equations for the two unknowns, C_3 and C_4.

$$\begin{cases} A = -C_3e^{-i\phi}\sin\dfrac{\theta}{2} + C_4\cos\dfrac{\theta}{2} \\ B = C_3\cos\dfrac{\theta}{2} + C_4e^{i\phi}\sin\dfrac{\theta}{2} \end{cases}$$

Eliminate C_3 by multiplying both sides of the first equation by $\cos(\theta/2)$ and multiplying both sides of the second equation by $e^{-i\phi}\sin(\theta/2)$

$$\begin{cases} A\cos\dfrac{\theta}{2} = -C_3e^{-i\phi}\cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2} + C_4\cos^2\dfrac{\theta}{2} \\ Be^{-i\phi}\sin\dfrac{\theta}{2} = C_3e^{-i\phi}\cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2} + C_4\sin^2\dfrac{\theta}{2} \end{cases}$$

and then adding the respective sides of these equations.

$$A\cos\frac{\theta}{2} + Be^{-i\phi}\sin\frac{\theta}{2} = C_4\cos^2\frac{\theta}{2} + C_4\sin^2\frac{\theta}{2}$$

$$= C_4$$

Eliminate C_4 by multiplying both sides of the first equation by $e^{i\phi}\sin(\theta/2)$ and multiplying both sides of the second equation by $\cos(\theta/2)$

$$\begin{cases} Ae^{i\phi}\sin\dfrac{\theta}{2} = -C_3\sin^2\dfrac{\theta}{2} + C_4e^{i\phi}\cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2} \\ B\cos\dfrac{\theta}{2} = C_3\cos^2\dfrac{\theta}{2} + C_4e^{i\phi}\cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2} \end{cases}$$

and then subtracting the respective sides of these equations.

$$Ae^{i\phi}\sin\frac{\theta}{2} - B\cos\frac{\theta}{2} = -C_3\sin^2\frac{\theta}{2} - C_3\cos^2\frac{\theta}{2}$$

$$Ae^{i\phi}\sin\frac{\theta}{2} - B\cos\frac{\theta}{2} = -C_3$$

As a result, measuring S_r on a spin-1/2 particle in the general state $\chi = \begin{bmatrix} A \\ B \end{bmatrix}$ yields values of

$$\frac{\hbar}{2} \quad \text{with probability} \quad |C_4|^2 = \left|A\cos\frac{\theta}{2} + Be^{-i\phi}\sin\frac{\theta}{2}\right|^2$$

$$-\frac{\hbar}{2} \quad \text{with probability} \quad |C_3|^2 = \left|Ae^{i\phi}\sin\frac{\theta}{2} - B\cos\frac{\theta}{2}\right|^2.$$

Now consider the sequential spin measurements: In the beginning there's an ensemble of spin-1/2 particles with spin up along $\boldsymbol{a}$. The initial state of these particles is

$$\chi_+^{(a)} = \begin{bmatrix} \cos\frac{\theta_a}{2} \\ e^{i\phi_a}\sin\frac{\theta_a}{2} \end{bmatrix},$$

where ϕ_a and θ_a are the angles of $\boldsymbol{a}$ in a spherical coordinate system. Orient the polar (z-) axis in the direction of $\boldsymbol{a}$ in order to replace θ_a with zero and simplify this initial state.

$$\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

As a result of measuring S_b, the spin angular momentum along $\boldsymbol{b}$, at time t_1 the values obtained are

$$\begin{cases} \dfrac{\hbar}{2} \quad \text{with probability} \quad \left|1\cdot\cos\dfrac{\theta_{ab}}{2} + 0\cdot e^{-i\alpha}\sin\dfrac{\theta_{ab}}{2}\right|^2 = \cos^2\dfrac{\theta_{ab}}{2} \\ -\dfrac{\hbar}{2} \quad \text{with probability} \quad \left|1\cdot e^{i\alpha}\sin\dfrac{\theta_{ab}}{2} - 0\cdot\cos\dfrac{\theta_{ab}}{2}\right|^2 = \sin^2\dfrac{\theta_{ab}}{2} \end{cases}$$

with the corresponding new states being

$$\begin{cases} \chi_+^{(b)} = \begin{bmatrix} \cos\frac{\theta_{ab}}{2} \\ \\ e^{i\alpha}\sin\frac{\theta_{ab}}{2} \end{bmatrix} \\ \\ \chi_-^{(b)} = \begin{bmatrix} -e^{-i\alpha}\sin\frac{\theta_{ab}}{2} \\ \\ \cos\frac{\theta_{ab}}{2} \end{bmatrix} \end{cases},$$

respectively, where θ_{ab} is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Orient the polar axis in the direction of $\boldsymbol{b}$ now to replace θ_{ab} with zero and simplify the states.

$$\begin{cases} \chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \\ \chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

Measuring S_c, the spin angular momentum along $\boldsymbol{c}$, at time t_2 on each of the particles in these states results in the following outcomes,

$$\begin{cases} \chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{cases} \dfrac{\hbar}{2} \quad \text{with probability} \quad \left|1\cdot\cos\dfrac{\theta_{bc}}{2} + 0\cdot e^{-i\beta}\sin\dfrac{\theta_{bc}}{2}\right|^2 = \cos^2\dfrac{\theta_{bc}}{2} \qquad \text{New state: } \chi_+^{(c)} = \begin{bmatrix} \cos\frac{\theta_{bc}}{2} \\ \\ e^{i\beta}\sin\frac{\theta_{bc}}{2} \end{bmatrix} \quad \textbf{Saved} \\ \\ -\dfrac{\hbar}{2} \quad \text{with probability} \quad \left|1\cdot e^{i\beta}\sin\dfrac{\theta_{bc}}{2} - 0\cdot\cos\dfrac{\theta_{bc}}{2}\right|^2 = \sin^2\dfrac{\theta_{bc}}{2} \qquad \text{New state: } \chi_-^{(c)} = \begin{bmatrix} -e^{-i\beta}\sin\frac{\theta_{bc}}{2} \\ \\ \cos\frac{\theta_{bc}}{2} \end{bmatrix} \end{cases} \\ \\ \chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{cases} \dfrac{\hbar}{2} \quad \text{with probability} \quad \left|0\cdot\cos\dfrac{\theta_{bc}}{2} + 1\cdot e^{-i\beta}\sin\dfrac{\theta_{bc}}{2}\right|^2 = \sin^2\dfrac{\theta_{bc}}{2} \qquad \text{New state: } \chi_+^{(c)} = \begin{bmatrix} \cos\frac{\theta_{bc}}{2} \\ \\ e^{i\beta}\sin\frac{\theta_{bc}}{2} \end{bmatrix} \quad \textbf{Saved} \\ \\ -\dfrac{\hbar}{2} \quad \text{with probability} \quad \left|0\cdot e^{i\beta}\sin\dfrac{\theta_{bc}}{2} - 1\cdot\cos\dfrac{\theta_{bc}}{2}\right|^2 = \cos^2\dfrac{\theta_{bc}}{2} \qquad \text{New state: } \chi_-^{(c)} = \begin{bmatrix} -e^{-i\beta}\sin\frac{\theta_{bc}}{2} \\ \\ \cos\frac{\theta_{bc}}{2} \end{bmatrix} \end{cases} \end{cases},$$

where θ_{bc} is the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$. Consequently, the probability of measuring $\hbar/2$ followed by $\hbar/2$ is

$$P_{++} = \cos^2\frac{\theta_{ab}}{2}\cos^2\frac{\theta_{bc}}{2},$$

the probability of measuring $\hbar/2$ followed by $-\hbar/2$ is

$$P_{+-} = \cos^2\frac{\theta_{ab}}{2}\sin^2\frac{\theta_{bc}}{2},$$

the probability of measuring $-\hbar/2$ followed by $\hbar/2$ is

$$P_{-+} = \sin^2\frac{\theta_{ab}}{2}\sin^2\frac{\theta_{bc}}{2},$$

and the probability of measuring $-\hbar/2$ followed by $-\hbar/2$ is

$$P_{--} = \sin^2\frac{\theta_{ab}}{2}\cos^2\frac{\theta_{bc}}{2}.$$

Save the particles with spin up after the measurement of S_c in a subensemble. Assuming there are N particles in the ensemble initially, the fraction of particles in the subensemble that had spin up along $\boldsymbol{b}$ is

$$\begin{aligned}
\text{fraction} &= \frac{\text{\# of particles with spin up along } \boldsymbol{c} \text{ that had spin up along } \boldsymbol{b}}{\text{\# of particles with spin up along } \boldsymbol{c}} \\
&= \frac{NP_{++}}{NP_{++} + NP_{-+}} \\
&= \frac{1}{1 + \frac{P_{-+}}{P_{++}}} \\
&= \frac{1}{1 + \frac{\sin^2\frac{\theta_{ab}}{2}\sin^2\frac{\theta_{bc}}{2}}{\cos^2\frac{\theta_{ab}}{2}\cos^2\frac{\theta_{bc}}{2}}} \\
&= \frac{1}{1 + \tan^2\frac{\theta_{ab}}{2}\tan^2\frac{\theta_{bc}}{2}} \\
&= \left(1 + \tan^2\frac{\theta_{ab}}{2}\tan^2\frac{\theta_{bc}}{2}\right)^{-1}.
\end{aligned}$$