Problem 4.72

Consider a particle with charge q, mass m, and spin s, in a uniform magnetic field $\mathbf{B}_0$. The vector potential can be chosen as

$$\mathbf{A} = -\frac{1}{2}\mathbf{r}\times\mathbf{B}_0.$$

(a) Verify that this vector potential produces a uniform magnetic field $\mathbf{B}_0$.

(b) Show that the Hamiltonian can be written

$$H = \frac{p^2}{2\,m} + q\,\varphi - \mathbf{B}_0\cdot(\gamma_o\,\mathbf{L} + \gamma\mathbf{S}) + \frac{q^2}{8m}\left[r^2\,B_0^2 - (\mathbf{r}\cdot\mathbf{B}_0)^2\right], \tag{4.230}$$

where $\gamma_o = q/2m$ is the gyromagnetic ratio for orbital motion. *Note*: The term linear in $\mathbf{B}_0$ makes it energetically favorable for the magnetic moments (orbital and spin) to align with the magnetic field; this is the origin of **paramagnetism** in materials. The term quadratic in $\mathbf{B}_{\mathbf{0}}$ leads to the opposite effect: **diamagnetism**.[75]

[TYPO: Put a comma between "obvious" and "but." Unbold the 0.]

Solution

Part (a)

The vector potential is defined by

$$\begin{aligned}
\mathbf{B} &= \nabla\times\mathbf{A}\\
&= \nabla\times\left(-\frac{1}{2}\mathbf{r}\times\mathbf{B}_0\right)\\
&= \left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\times\left[-\frac{1}{2}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k x_k\right)\times\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l B_{0l}\right)\right]\\
&= \left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\times\left[-\frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_k\times\boldsymbol{\delta}_l)x_k B_{0l}\right]\\
&= \left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\times\left(-\frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\varepsilon_{kln}\boldsymbol{\delta}_n x_k B_{0l}\right)\\
&= -\frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\varepsilon_{kln}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_n)\frac{\partial}{\partial x_j}(x_k B_{0l}).
\end{aligned}$$

[75] That's not obvious but we'll prove it in Chapter 7.

Continue the simplification.

$$\begin{aligned}
\mathbf{B} &= -\frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{o=1}^{3}\varepsilon_{kln}\varepsilon_{jno}\boldsymbol{\delta}_o\frac{\partial}{\partial x_j}(x_k B_{0l}) \\
&= -\frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{o=1}^{3}\varepsilon_{kln}\varepsilon_{ojn}\boldsymbol{\delta}_o B_{0l}\frac{\partial x_k}{\partial x_j} \\
&= -\frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{o=1}^{3}(\delta_{ko}\delta_{lj}-\delta_{kj}\delta_{lo})\boldsymbol{\delta}_o B_{0l}\delta_{jk} \\
&= -\frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{o=1}^{3}(\delta_{ko}\delta_{lk}-\delta_{kk}\delta_{lo})\boldsymbol{\delta}_o B_{0l} \\
&= -\frac{1}{2}\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{o=1}^{3}\delta_{ko}\delta_{lk}\boldsymbol{\delta}_o B_{0l}-\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{o=1}^{3}\delta_{kk}\delta_{lo}\boldsymbol{\delta}_o B_{0l}\right) \\
&= -\frac{1}{2}\left[\sum_{l=1}^{3}\sum_{o=1}^{3}\delta_{lo}\boldsymbol{\delta}_o B_{0l}-\left(\sum_{k=1}^{3}\delta_{kk}\right)\sum_{l=1}^{3}\sum_{o=1}^{3}\delta_{lo}\boldsymbol{\delta}_o B_{0l}\right] \\
&= -\frac{1}{2}\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l B_{0l}-3\sum_{l=1}^{3}\boldsymbol{\delta}_l B_{0l}\right) \\
&= -\frac{1}{2}\left(-2\sum_{l=1}^{3}\boldsymbol{\delta}_l B_{0l}\right) \\
&= \sum_{l=1}^{3}\boldsymbol{\delta}_l B_{0l} \\
&= \mathbf{B}_0
\end{aligned}$$

$\mathbf{A} = -\frac{1}{2}\mathbf{r}\times\mathbf{B}_0$ does indeed give a uniform magnetic field $\mathbf{B} = \mathbf{B}_0$.

Part (b)

The Hamiltonian for a particle of charge q and mass m in electromagnetic fields is given in Equation 4.188 on page 181. Add this to the Hamiltonian for a particle with spin (Equation 4.157 on page 172) to obtain the total. Use the fact that the magnetic dipole moment is proportional to spin angular momentum: $\boldsymbol{\mu} = \gamma \mathbf{S}$.

$$H = \underbrace{\frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\varphi}_{\substack{\text{energy associated with} \\ \text{charged moving particle} \\ \text{in EM fields}}} + \underbrace{(-\boldsymbol{\mu} \cdot \mathbf{B})}_{\substack{\text{energy associated with} \\ \text{torque on magnetic dipole} \\ \text{resulting from spin}}}$$

$$= \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\varphi - (\gamma \mathbf{S}) \cdot \mathbf{B}$$

$$= \frac{1}{2m}(\mathbf{p} - q\mathbf{A}) \cdot (\mathbf{p} - q\mathbf{A}) + q\varphi - \gamma \mathbf{S} \cdot \mathbf{B}$$

$$= \frac{1}{2m}(\mathbf{p} \cdot \mathbf{p} - q\mathbf{p} \cdot \mathbf{A} - q\mathbf{A} \cdot \mathbf{p} + q^2 \mathbf{A} \cdot \mathbf{A}) + q\varphi - \gamma \mathbf{B} \cdot \mathbf{S}$$

$$= \frac{\mathbf{p} \cdot \mathbf{p}}{2m} + q\varphi - \gamma \mathbf{B} \cdot \mathbf{S} - \frac{q}{2m}(\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}) + \frac{q^2}{2m} \mathbf{A} \cdot \mathbf{A}$$

$$= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{q}{2m}\left[(-i\hbar\nabla) \cdot \left(-\frac{1}{2}\mathbf{r} \times \mathbf{B}_0\right) + \left(-\frac{1}{2}\mathbf{r} \times \mathbf{B}_0\right) \cdot (-i\hbar\nabla)\right] + \frac{q^2}{2m}\left[\left(-\frac{1}{2}\mathbf{r} \times \mathbf{B}_0\right) \cdot \left(-\frac{1}{2}\mathbf{r} \times \mathbf{B}_0\right)\right]$$

$$= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r} \times \mathbf{B}_0) + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla] + \frac{q^2}{8m}[(\mathbf{r} \times \mathbf{B}_0) \cdot (\mathbf{r} \times \mathbf{B}_0)]$$

$$= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r} \times \mathbf{B}_0) + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla] + \frac{q^2}{8m}\left\{\left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j x_j\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_{0k}\right)\right] \cdot \left[\left(\sum_{l=1}^{3} \boldsymbol{\delta}_l x_l\right) \times \left(\sum_{n=1}^{3} \boldsymbol{\delta}_n B_{0n}\right)\right]\right\}$$

$$= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r} \times \mathbf{B}_0) + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla] + \frac{q^2}{8m}\left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) x_j B_{0k}\right] \cdot \left[\sum_{l=1}^{3}\sum_{n=1}^{3} (\boldsymbol{\delta}_l \times \boldsymbol{\delta}_n) x_l B_{0n}\right]$$

Continue the simplification.

$$
\begin{aligned}
H &= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r}\times\mathbf{B}_0) + (\mathbf{r}\times\mathbf{B}_0)\cdot\nabla] + \frac{q^2}{8m}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{o=1}^{3}\varepsilon_{jko}\boldsymbol{\delta}_o x_j B_{0k}\right)\cdot\left(\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{t=1}^{3}\varepsilon_{lnt}\boldsymbol{\delta}_t x_l B_{0n}\right) \\
&= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r}\times\mathbf{B}_0) + (\mathbf{r}\times\mathbf{B}_0)\cdot\nabla] + \frac{q^2}{8m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{o=1}^{3}\sum_{t=1}^{3}\varepsilon_{jko}\varepsilon_{lnt}(\boldsymbol{\delta}_o\cdot\boldsymbol{\delta}_t)x_j x_l B_{0k}B_{0n} \\
&= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r}\times\mathbf{B}_0) + (\mathbf{r}\times\mathbf{B}_0)\cdot\nabla] + \frac{q^2}{8m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{o=1}^{3}\sum_{t=1}^{3}\varepsilon_{jko}\varepsilon_{lnt}\delta_{ot}x_j x_l B_{0k}B_{0n} \\
&= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r}\times\mathbf{B}_0) + (\mathbf{r}\times\mathbf{B}_0)\cdot\nabla] + \frac{q^2}{8m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\sum_{o=1}^{3}\varepsilon_{jko}\varepsilon_{lno}x_j x_l B_{0k}B_{0n} \\
&= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r}\times\mathbf{B}_0) + (\mathbf{r}\times\mathbf{B}_0)\cdot\nabla] + \frac{q^2}{8m}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}(\delta_{jl}\delta_{kn}-\delta_{jn}\delta_{kl})x_j x_l B_{0k}B_{0n} \\
&= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r}\times\mathbf{B}_0) + (\mathbf{r}\times\mathbf{B}_0)\cdot\nabla] + \frac{q^2}{8m}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\delta_{jl}\delta_{kn}x_j x_l B_{0k}B_{0n} - \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{n=1}^{3}\delta_{jn}\delta_{kl}x_j x_l B_{0k}B_{0n}\right) \\
&= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r}\times\mathbf{B}_0) + (\mathbf{r}\times\mathbf{B}_0)\cdot\nabla] + \frac{q^2}{8m}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jl}x_j x_l B_{0k}B_{0k} - \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{kl}x_j x_l B_{0k}B_{0j}\right) \\
&= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r}\times\mathbf{B}_0) + (\mathbf{r}\times\mathbf{B}_0)\cdot\nabla] + \frac{q^2}{8m}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}x_j x_j B_{0k}B_{0k} - \sum_{j=1}^{3}\sum_{k=1}^{3}x_j x_k B_{0k}B_{0j}\right)
\end{aligned}
$$

As a result,

$$\begin{aligned}
H &= \frac{p^2}{2m} + q\varphi - \gamma\mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r} \times \mathbf{B}_0) + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla] + \frac{q^2}{8m}\left[\left(\sum_{j=1}^3 x_j^2\right)\left(\sum_{k=1}^3 B_{0k}^2\right) - \left(\sum_{j=1}^3 x_j B_{0j}\right)\left(\sum_{k=1}^3 x_k B_{0k}\right)\right] \\
&= \frac{p^2}{2m} + q\varphi - \gamma\mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r} \times \mathbf{B}_0) + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla] + \frac{q^2}{8m}[r^2 B_0^2 - (\mathbf{r} \cdot \mathbf{B}_0)(\mathbf{r} \cdot \mathbf{B}_0)] \\
&= \frac{p^2}{2m} + q\varphi - \gamma\mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m}[\nabla \cdot (\mathbf{r} \times \mathbf{B}_0) + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla] + \frac{q^2}{8m}[r^2 B_0^2 - (\mathbf{r} \cdot \mathbf{B}_0)^2]. \qquad (1)
\end{aligned}$$

In order to simplify the term with nabla operators, let it act on a test function f.

$$\begin{aligned}
[\nabla \cdot (\mathbf{r} \times \mathbf{B}_0) + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla]f &= \nabla \cdot (\mathbf{r} \times \mathbf{B}_0)f + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla f \\
&= \left(\sum_{j=1}^3 \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left[\left(\sum_{k=1}^3 \boldsymbol{\delta}_k x_k\right) \times \left(\sum_{l=1}^3 \boldsymbol{\delta}_l B_{0l}\right)\right] f + \left[\left(\sum_{n=1}^3 \boldsymbol{\delta}_n x_n\right) \times \left(\sum_{o=1}^3 \boldsymbol{\delta}_o B_{0o}\right)\right] \cdot \left(\sum_{t=1}^3 \boldsymbol{\delta}_t \frac{\partial}{\partial x_t}\right) f \\
&= \left(\sum_{j=1}^3 \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left[\sum_{k=1}^3 \sum_{l=1}^3 (\boldsymbol{\delta}_k \times \boldsymbol{\delta}_l) x_k B_{0l} f\right] + \left[\sum_{n=1}^3 \sum_{o=1}^3 (\boldsymbol{\delta}_n \times \boldsymbol{\delta}_o) x_n B_{0o}\right] \cdot \left(\sum_{t=1}^3 \boldsymbol{\delta}_t \frac{\partial f}{\partial x_t}\right) \\
&= \left(\sum_{j=1}^3 \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left(\sum_{k=1}^3 \sum_{l=1}^3 \sum_{u=1}^3 \varepsilon_{klu} \boldsymbol{\delta}_u x_k B_{0l} f\right) + \left(\sum_{n=1}^3 \sum_{o=1}^3 \sum_{v=1}^3 \varepsilon_{nov} \boldsymbol{\delta}_v x_n B_{0o}\right) \cdot \left(\sum_{t=1}^3 \boldsymbol{\delta}_t \frac{\partial f}{\partial x_t}\right) \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{u=1}^3 \varepsilon_{klu} (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_u) \frac{\partial}{\partial x_j}(x_k B_{0l} f) + \sum_{n=1}^3 \sum_{o=1}^3 \sum_{t=1}^3 \sum_{v=1}^3 \varepsilon_{nov} (\boldsymbol{\delta}_v \cdot \boldsymbol{\delta}_t) x_n B_{0o} \frac{\partial f}{\partial x_t} \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{u=1}^3 \varepsilon_{klu} \delta_{ju} \frac{\partial}{\partial x_j}(x_k B_{0l} f) + \sum_{n=1}^3 \sum_{o=1}^3 \sum_{t=1}^3 \sum_{v=1}^3 \varepsilon_{nov} \delta_{vt} x_n B_{0o} \frac{\partial f}{\partial x_t}
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
[\nabla\cdot(\mathbf{r}\times\mathbf{B}_0)+(\mathbf{r}\times\mathbf{B}_0)\cdot\nabla]f &= \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{klj}\frac{\partial}{\partial x_j}(x_kB_{0l}f)+\sum_{n=1}^3\sum_{o=1}^3\sum_{t=1}^3\varepsilon_{not}x_nB_{0o}\frac{\partial f}{\partial x_t} \\
&= \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{jkl}\frac{\partial}{\partial x_j}(x_kB_{0l}f)+\sum_{t=1}^3\sum_{n=1}^3\sum_{o=1}^3\varepsilon_{tno}x_nB_{0o}\frac{\partial f}{\partial x_t} \\
&= \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{jkl}B_{0l}\frac{\partial}{\partial x_j}(x_kf)+\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{jkl}x_kB_{0l}\frac{\partial f}{\partial x_j} \\
&= \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{jkl}B_{0l}\left[\frac{\partial}{\partial x_j}(x_kf)+x_k\frac{\partial f}{\partial x_j}\right] \\
&= \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{jkl}B_{0l}\left(\frac{\partial x_k}{\partial x_j}f+x_k\frac{\partial f}{\partial x_j}+x_k\frac{\partial f}{\partial x_j}\right) \\
&= \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{jkl}B_{0l}\left(\delta_{jk}f+2x_k\frac{\partial f}{\partial x_j}\right) \\
&= \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{jkl}B_{0l}\delta_{jk}f+2\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{jkl}B_{0l}x_k\frac{\partial f}{\partial x_j} \\
&= \sum_{k=1}^3\sum_{l=1}^3\varepsilon_{kkl}B_{0l}f+2\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3(-\varepsilon_{kjl})B_{0l}x_k\frac{\partial f}{\partial x_j} \\
&= \sum_{k=1}^3\sum_{l=1}^3(0)B_{0l}f-2\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\sum_{o=1}^3\varepsilon_{kjl}\delta_{ol}B_{0o}x_k\frac{\partial f}{\partial x_j} \\
&= -2\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\sum_{o=1}^3\varepsilon_{kjl}(\boldsymbol{\delta}_o\cdot\boldsymbol{\delta}_l)B_{0o}x_k\frac{\partial f}{\partial x_j} \\
&= -2\left(\sum_{o=1}^3\boldsymbol{\delta}_oB_{0o}\right)\cdot\left(\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\varepsilon_{kjl}\boldsymbol{\delta}_lx_k\frac{\partial f}{\partial x_j}\right) \\
&= -2\left(\sum_{o=1}^3\boldsymbol{\delta}_oB_{0o}\right)\cdot\left[\sum_{j=1}^3\sum_{k=1}^3(\boldsymbol{\delta}_k\times\boldsymbol{\delta}_j)x_k\frac{\partial f}{\partial x_j}\right]
\end{aligned}$$

Consequently,

$$\begin{aligned}[\nabla \cdot (\mathbf{r} \times \mathbf{B}_0) + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla] f &= -2 \left(\sum_{o=1}^{3} \boldsymbol{\delta}_o B_{0o} \right) \cdot \left[\sum_{j=1}^{3} \sum_{k=1}^{3} (\boldsymbol{\delta}_k \times \boldsymbol{\delta}_j) x_k \frac{\partial}{\partial x_j} \right] f \\ &= -2 \left(\sum_{o=1}^{3} \boldsymbol{\delta}_o B_{0o} \right) \cdot \left[\left(\sum_{k=1}^{3} \boldsymbol{\delta}_k x_k \right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j} \right) \right] f \\ &= -2\mathbf{B}_0 \cdot (\mathbf{r} \times \nabla) f,\end{aligned}$$

which makes equation (1) become

$$\begin{aligned}H &= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m} [\nabla \cdot (\mathbf{r} \times \mathbf{B}_0) + (\mathbf{r} \times \mathbf{B}_0) \cdot \nabla] + \frac{q^2}{8m} [r^2 B_0^2 - (\mathbf{r} \cdot \mathbf{B}_0)^2] \qquad (1) \\ &= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{i\hbar q}{4m} [-2\mathbf{B}_0 \cdot (\mathbf{r} \times \nabla)] + \frac{q^2}{8m} [r^2 B_0^2 - (\mathbf{r} \cdot \mathbf{B}_0)^2] \\ &= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \frac{q}{2m} \mathbf{B}_0 \cdot [\mathbf{r} \times (-i\hbar \nabla)] + \frac{q^2}{8m} [r^2 B_0^2 - (\mathbf{r} \cdot \mathbf{B}_0)^2] \\ &= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \gamma_o \mathbf{B}_0 \cdot (\mathbf{r} \times \mathbf{p}) + \frac{q^2}{8m} [r^2 B_0^2 - (\mathbf{r} \cdot \mathbf{B}_0)^2] \\ &= \frac{p^2}{2m} + q\varphi - \gamma \mathbf{B}_0 \cdot \mathbf{S} - \gamma_o \mathbf{B}_0 \cdot \mathbf{L} + \frac{q^2}{8m} [r^2 B_0^2 - (\mathbf{r} \cdot \mathbf{B}_0)^2] \\ &= \frac{p^2}{2m} + q\varphi - \mathbf{B}_0 \cdot (\gamma \mathbf{S} + \gamma_o \mathbf{L}) + \frac{q^2}{8m} [r^2 B_0^2 - (\mathbf{r} \cdot \mathbf{B}_0)^2].\end{aligned}$$

Therefore,

$$H = \frac{p^2}{2m} + q\varphi - \mathbf{B}_0 \cdot (\gamma_o \mathbf{L} + \gamma \mathbf{S}) + \frac{q^2}{8m} [r^2 B_0^2 - (\mathbf{r} \cdot \mathbf{B}_0)^2],$$

where $\gamma_o = q/2m$ is the gyromagnetic ratio for orbital motion.