Problem 4.73

Example 4.4, couched in terms of forces, was a quasi-classical explanation for the Stern–Gerlach effect. Starting from the Hamiltonian for a neutral, spin-1/2 particle traveling through the magnetic field given by Equation 4.169,

$$H = \frac{p^2}{2m} - \gamma\,\mathbf{B}\cdot\mathbf{S},$$

use the generalized Ehrenfest theorem (Equation 3.73) to show that

$$m\,\frac{d^2}{dt^2}\,\langle z\rangle = \gamma\,\alpha\,\langle S_z\rangle.$$

Comment: Equation 4.170 is therefore a correct quantum-mechanical statement, with the understanding that the quantities refer to expectation values.

Solution

Equation 3.73 is on page 110 and holds for any observable Q.

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\left\langle\left[\hat{H},\hat{Q}\right]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle \tag{3.73}$$

Recall also the commutator identities shown in Exercise 3.14 on page 108.

$$\left[\hat{A}+\hat{B},\hat{C}\right] = \left[\hat{A},\hat{C}\right] + \left[\hat{B},\hat{C}\right]$$

$$\left[\hat{A}\hat{B},\hat{C}\right] = \hat{A}\left[\hat{B},\hat{C}\right] + \left[\hat{A},\hat{C}\right]\hat{B}$$

Recall also the commutation relations for the components of $\mathbf{r}$ and $\mathbf{p}$ shown in Exercise 4.1 on page 132.

$$[r_j,p_k] = -[p_j,r_k] = i\hbar\delta_{jk}$$

$$[r_j,r_k] = [p_j,p_k] = 0$$

Start by using Equation 3.73 to find the time derivative of the expectation value of z.

$$\begin{aligned}
\frac{d}{dt}\langle z\rangle &= \frac{i}{\hbar}\left\langle\left[\hat{H},z\right]\right\rangle + \overbrace{\left\langle\frac{\partial z}{\partial t}\right\rangle}^{=0} \\
&= \frac{i}{\hbar}\left\langle\left[\frac{p^2}{2m} - \gamma\,\mathbf{B}\cdot\mathbf{S}, z\right]\right\rangle \\
&= \frac{i}{\hbar}\left\langle\left[\frac{p^2}{2m}, z\right] - [\gamma\,\mathbf{B}\cdot\mathbf{S}, z]\right\rangle
\end{aligned}$$

Bring the constants in front. Note that the magnetic field in Equation 4.169 on page 174 is $\mathbf{B}(x,y,z) = -\alpha x\hat{\mathbf{x}} + (B_0 + \alpha z)\hat{\mathbf{z}}$.

$$\begin{aligned}
\frac{d}{dt}\langle z\rangle &= \frac{i}{\hbar}\left\langle \frac{1}{2m}\left[p^2, z\right] - \gamma\left[\mathbf{B}\cdot\mathbf{S}, z\right]\right\rangle \\
&= \frac{i}{2m\hbar}\left\langle \left[p^2, z\right]\right\rangle - \frac{i\gamma}{\hbar}\left\langle [\mathbf{B}\cdot\mathbf{S}, z]\right\rangle \\
&= \frac{i}{2m\hbar}\left\langle \left[p_x^2 + p_y^2 + p_z^2, z\right]\right\rangle - \frac{i\gamma}{\hbar}\left\langle [\mathbf{B}\cdot\mathbf{S}, z]\right\rangle \\
&= \frac{i}{2m\hbar}\left\langle [p_x^2, z] + [p_y^2, z] + [p_z^2, z]\right\rangle - \frac{i\gamma}{\hbar}\left\langle [\mathbf{B}\cdot\mathbf{S}, z]\right\rangle \\
&= \frac{i}{2m\hbar}\left\langle p_x[p_x, z] + [p_x, z]p_x + p_y[p_y, z] + [p_y, z]p_y + p_z[p_z, z] + [p_z, z]p_z\right\rangle - \frac{i\gamma}{\hbar}\left\langle [\mathbf{B}\cdot\mathbf{S}, z]\right\rangle \\
&= \frac{i}{2m\hbar}\left\langle p_x(0) + (0)p_x + p_y(0) + (0)p_y + p_z(-i\hbar) + (-i\hbar)p_z\right\rangle - \frac{i\gamma}{\hbar}\left\langle [\mathbf{B}\cdot\mathbf{S}, z]\right\rangle \\
&= \frac{i}{2m\hbar}\left\langle -2i\hbar p_z\right\rangle - \frac{i\gamma}{\hbar}\left\langle [\mathbf{B}\cdot\mathbf{S}, z]\right\rangle \\
&= \frac{1}{m}\langle p_z\rangle - \frac{i\gamma}{\hbar}\langle \mathbf{\Psi}\,|\,[\mathbf{B}\cdot\mathbf{S}, z]\,|\,\mathbf{\Psi}\rangle \\
&= \frac{1}{m}\langle p_z\rangle - \frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}} \mathbf{\Psi}^\dagger[\mathbf{B}\cdot\mathbf{S}, z]\mathbf{\Psi}\, d\mathcal{V} \\
&= \frac{1}{m}\langle p_z\rangle - \frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}} \mathbf{\Psi}^\dagger[(\mathbf{B}\cdot\mathbf{S})z - z(\mathbf{B}\cdot\mathbf{S})]\mathbf{\Psi}\, d\mathcal{V} \\
&= \frac{1}{m}\langle p_z\rangle - \frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}} (\Psi_+\chi_+ + \Psi_-\chi_-)^\dagger[(\mathbf{B}\cdot\mathbf{S})z - z(\mathbf{B}\cdot\mathbf{S})](\Psi_+\chi_+ + \Psi_-\chi_-)\, d\mathcal{V} \\
&= \frac{1}{m}\langle p_z\rangle - \frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix}^\dagger [(\mathbf{B}\cdot\mathbf{S})z - z(\mathbf{B}\cdot\mathbf{S})] \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix} d\mathcal{V} \\
&= \frac{1}{m}\langle p_z\rangle - \frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix}^\dagger \{[-\alpha x\mathsf{S}_x + (B_0 + \alpha z)\mathsf{S}_z]z - z[-\alpha x\mathsf{S}_x + (B_0 + \alpha z)\mathsf{S}_z]\} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix} d\mathcal{V}
\end{aligned}$$

Continue the simplification.

$$\frac{d}{dt}\langle z\rangle = \frac{1}{m}\langle p_z\rangle - \frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix}^\dagger \left\{\frac{\hbar}{2}\begin{bmatrix}B_0+\alpha z & -\alpha x \\ -\alpha x & -B_0-\alpha z\end{bmatrix} z - z\frac{\hbar}{2}\begin{bmatrix}B_0+\alpha z & -\alpha x \\ -\alpha x & -B_0-\alpha z\end{bmatrix}\right\} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix} d\mathcal{V}$$

$$= \frac{1}{m}\langle p_z\rangle - \frac{i\gamma}{2}\iiint\limits_{\text{all space}} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix}^\dagger \left\{\begin{bmatrix}B_0+\alpha z & -\alpha x \\ -\alpha x & -B_0-\alpha z\end{bmatrix} z - z\begin{bmatrix}B_0+\alpha z & -\alpha x \\ -\alpha x & -B_0-\alpha z\end{bmatrix}\right\} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix} d\mathcal{V}$$

$$= \frac{1}{m}\langle p_z\rangle - \frac{i\gamma}{2}\iiint\limits_{\text{all space}} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix}^\dagger \left\{\begin{bmatrix}B_0 z+\alpha z^2 & -\alpha xz \\ -\alpha xz & -B_0 z-\alpha z^2\end{bmatrix} - \begin{bmatrix}B_0 z+\alpha z^2 & -\alpha xz \\ -\alpha xz & -B_0 z-\alpha z^2\end{bmatrix}\right\} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix} d\mathcal{V}$$

$$= \frac{1}{m}\langle p_z\rangle - \underbrace{\frac{i\gamma}{2}\iiint\limits_{\text{all space}} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix}^\dagger \begin{bmatrix}0 & 0 \\ 0 & 0\end{bmatrix} \begin{bmatrix}\Psi_+ \\ \Psi_-\end{bmatrix} d\mathcal{V}}_{=0}$$

$$= \frac{1}{m}\langle p_z\rangle$$

Multiply both sides by m, differentiate both sides with respect to t, and use Equation 3.73 again.

$$m\frac{d}{dt}\langle z\rangle = \langle p_z\rangle$$

$$\frac{d}{dt}\left(m\frac{d}{dt}\langle z\rangle\right) = \frac{d}{dt}\langle p_z\rangle$$

$$m\frac{d^2}{dt^2}\langle z\rangle = \frac{i}{\hbar}\left\langle\left[\hat{H}, p_z\right]\right\rangle + \overbrace{\left\langle\frac{\partial p_z}{\partial t}\right\rangle}^{=0}$$

$$= \frac{i}{\hbar}\left\langle\left[\frac{p^2}{2m} - \gamma\,\mathbf{B}\cdot\mathbf{S}, p_z\right]\right\rangle$$

$$= \frac{i}{\hbar}\left\langle\left[\frac{p^2}{2m}, p_z\right] - [\gamma\,\mathbf{B}\cdot\mathbf{S}, p_z]\right\rangle$$

$$= \frac{i}{\hbar}\left\langle\frac{1}{2m}\left[p^2, p_z\right] - \gamma\left[\mathbf{B}\cdot\mathbf{S}, p_z\right]\right\rangle$$

$$= \frac{i}{\hbar}\left\langle\frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2, p_z\right] - \gamma\left[\mathbf{B}\cdot\mathbf{S}, p_z\right]\right\rangle$$

Continue the simplification, noting that the magnetic field in Equation 4.169 on page 174 is $\mathbf{B}(x,y,z) = -\alpha x\hat{\mathbf{x}} + (B_0 + \alpha z)\hat{\mathbf{z}}$.

$$m\frac{d^2}{dt^2}\langle z\rangle = \frac{i}{\hbar}\left\langle \frac{1}{2m}\left([p_x^2, p_z] + [p_y^2, p_z] + [p_z^2, p_z]\right) - \gamma\,[\mathbf{B}\cdot\mathbf{S}, p_z]\right\rangle$$

$$= \frac{i}{\hbar}\left\langle \frac{1}{2m}\left(p_x\,[p_x,p_z] + [p_x,p_z]\,p_x + p_y\,[p_y,p_z] + [p_y,p_z]\,p_y + p_z\,[p_z,p_z] + [p_z,p_z]\,p_z\right) - \gamma\,[\mathbf{B}\cdot\mathbf{S}, p_z]\right\rangle$$

$$= \frac{i}{\hbar}\left\langle \frac{1}{2m}[p_x(0) + (0)p_x + p_y(0) + (0)p_y + p_z(0) + (0)p_z] - \gamma\,[\mathbf{B}\cdot\mathbf{S}, p_z]\right\rangle$$

$$= -\frac{i\gamma}{\hbar}\langle[\mathbf{B}\cdot\mathbf{S}, p_z]\rangle$$

$$= -\frac{i\gamma}{\hbar}\langle\mathbf{\Psi}\mid[\mathbf{B}\cdot\mathbf{S}, p_z]\mid\mathbf{\Psi}\rangle$$

$$= -\frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}}\mathbf{\Psi}^\dagger[\mathbf{B}\cdot\mathbf{S}, p_z]\mathbf{\Psi}\,d\mathcal{V}$$

$$= -\frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}}\mathbf{\Psi}^\dagger[(\mathbf{B}\cdot\mathbf{S})p_z - p_z(\mathbf{B}\cdot\mathbf{S})]\mathbf{\Psi}\,d\mathcal{V}$$

$$= -\frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}}(\Psi_+\chi_+ + \Psi_-\chi_-)^\dagger[(\mathbf{B}\cdot\mathbf{S})p_z - p_z(\mathbf{B}\cdot\mathbf{S})](\Psi_+\chi_+ + \Psi_-\chi_-)\,d\mathcal{V}$$

$$= -\frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger[(\mathbf{B}\cdot\mathbf{S})p_z - p_z(\mathbf{B}\cdot\mathbf{S})]\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}d\mathcal{V}$$

$$= -\frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger\{[-\alpha x\mathsf{S}_x + (B_0+\alpha z)\mathsf{S}_z]p_z - p_z[-\alpha x\mathsf{S}_x + (B_0+\alpha z)\mathsf{S}_z]\}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}d\mathcal{V}$$

$$= -\frac{i\gamma}{\hbar}\iiint\limits_{\text{all space}}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger\left\{\frac{\hbar}{2}\begin{bmatrix}B_0+\alpha z & -\alpha x\\ -\alpha x & -B_0-\alpha z\end{bmatrix}p_z - p_z\frac{\hbar}{2}\begin{bmatrix}B_0+\alpha z & -\alpha x\\ -\alpha x & -B_0-\alpha z\end{bmatrix}\right\}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}d\mathcal{V}$$

$$= -\frac{i\gamma}{2}\iiint\limits_{\text{all space}}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger\left\{\begin{bmatrix}B_0+\alpha z & -\alpha x\\ -\alpha x & -B_0-\alpha z\end{bmatrix}\left(-i\hbar\frac{\partial}{\partial z}\right) - \left(-i\hbar\frac{\partial}{\partial z}\right)\begin{bmatrix}B_0+\alpha z & -\alpha x\\ -\alpha x & -B_0-\alpha z\end{bmatrix}\right\}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}d\mathcal{V}$$

$$= \frac{\hbar\gamma}{2}\iiint\limits_{\text{all space}}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger\left\{-\begin{bmatrix}B_0+\alpha z & -\alpha x\\ -\alpha x & -B_0-\alpha z\end{bmatrix}\frac{\partial}{\partial z} + \frac{\partial}{\partial z}\begin{bmatrix}B_0+\alpha z & -\alpha x\\ -\alpha x & -B_0-\alpha z\end{bmatrix}\right\}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}d\mathcal{V}$$

Therefore,

$$m\frac{d^2}{dt^2}\langle z\rangle = \frac{\hbar\gamma}{2}\iiint_{\text{all space}} \begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger \left\{\begin{bmatrix}-B_0-\alpha z & \alpha x\\ \alpha x & B_0+\alpha z\end{bmatrix}\begin{bmatrix}\frac{\partial\Psi_+}{\partial z}\\ \frac{\partial\Psi_-}{\partial z}\end{bmatrix} + \frac{\partial}{\partial z}\begin{bmatrix}B_0+\alpha z & -\alpha x\\ -\alpha x & -B_0-\alpha z\end{bmatrix}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}\right\} d\mathcal{V}$$

$$= \frac{\hbar\gamma}{2}\iiint_{\text{all space}} \begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger \left\{\begin{bmatrix}(-B_0-\alpha z)\frac{\partial\Psi_+}{\partial z}+\alpha x\frac{\partial\Psi_-}{\partial z}\\ \alpha x\frac{\partial\Psi_+}{\partial z}+(B_0+\alpha z)\frac{\partial\Psi_-}{\partial z}\end{bmatrix} + \frac{\partial}{\partial z}\begin{bmatrix}(B_0+\alpha z)\Psi_+ - \alpha x\Psi_-\\ -\alpha x\Psi_+ + (-B_0-\alpha z)\Psi_-\end{bmatrix}\right\} d\mathcal{V}$$

$$= \frac{\hbar\gamma}{2}\iiint_{\text{all space}} \begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger \left\{\begin{bmatrix}(-B_0-\alpha z)\frac{\partial\Psi_+}{\partial z}+\alpha x\frac{\partial\Psi_-}{\partial z}\\ \alpha x\frac{\partial\Psi_+}{\partial z}+(B_0+\alpha z)\frac{\partial\Psi_-}{\partial z}\end{bmatrix} + \begin{bmatrix}(B_0+\alpha z)\frac{\partial\Psi_+}{\partial z}+\alpha\Psi_+ - \alpha x\frac{\partial\Psi_-}{\partial z}\\ -\alpha x\frac{\partial\Psi_+}{\partial z} + (-B_0-\alpha z)\frac{\partial\Psi_-}{\partial z}-\alpha\Psi_-\end{bmatrix}\right\} d\mathcal{V}$$

$$= \frac{\hbar\gamma}{2}\iiint_{\text{all space}} \begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger \begin{bmatrix}\alpha\Psi_+\\ -\alpha\Psi_-\end{bmatrix} d\mathcal{V}$$

$$= \frac{\hbar\gamma\alpha}{2}\iiint_{\text{all space}} \begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger \begin{bmatrix}\Psi_+\\ -\Psi_-\end{bmatrix} d\mathcal{V}$$

$$= \frac{\hbar\gamma\alpha}{2}\iiint_{\text{all space}} \begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix}^\dagger \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\Psi_+\\ \Psi_-\end{bmatrix} d\mathcal{V}$$

$$= \gamma\alpha\iiint_{\text{all space}} (\Psi_+\chi_+ + \Psi_-\chi_-)^\dagger \mathsf{S}_z(\Psi_+\chi_+ + \Psi_-\chi_-)\, d\mathcal{V}$$

$$= \gamma\alpha\iiint_{\text{all space}} \mathbf{\Psi}^\dagger S_z\mathbf{\Psi}\, d\mathcal{V}$$

$$= \gamma\alpha\,\langle\mathbf{\Psi}\,|\,S_z\,|\,\mathbf{\Psi}\rangle$$

$$= \gamma\alpha\langle S_z\rangle.$$