Exercise 1.2.6

Suppose that the specific heat is a function of position and temperature, $c(x, u)$.

(a) Show that the heat energy per unit mass necessary to raise the temperature of a thin slice of thickness Δx from 0° to $u(x, t)$ is not $c(x)u(x, t)$, but instead $\int_0^u c(x, \bar{u})\, d\bar{u}$.

(b) Rederive the heat equation in this case. Show that (1.2.3) remains unchanged.

Solution

Part (a)

The heat capacity C is defined as the amount of thermal energy it takes to raise the temperature of some mass by one unit. In this exercise, however, it takes more or less energy to raise the temperature by one unit if the mass is at one temperature compared to another. Assume that at the temperature u_1 the mass has thermal energy U_1, and at the temperature u_2 the mass has thermal energy U_2. An approximate formula for the heat capacity is given by the difference quotient.

$$\frac{U_2 - U_1}{T_2 - T_1} \approx C$$

As u_2 gets closer and closer to u_1, the approximation to c becomes better and better. In the limit as u_2 approaches u_1, equality results.

$$\lim_{u_2 \to u_1} \frac{U_2 - U_1}{u_2 - u_1} = C$$

The left side is the first derivative of U with respect to u.

$$\frac{dU}{du} = C$$

To solve this differential equation, separate variables.

$$dU = C\, du$$

Integrate both sides.

$$\int_{U_1}^{U_2} dU = \int_{u_1}^{u_2} C(\bar{u})\, d\bar{u}$$

In this exercise the temperature goes from 0° to u, so the limits of the integral on the right side will be replaced. Solve the integral on the left side.

$$U_2 - U_1 = \int_0^u C(\bar{u})\, d\bar{u}$$

Let $\Delta U = U_2 - U_1$. It represents the change in thermal energy of the mass as a result of going from 0° to u.

$$\Delta U = \int_0^u C(\bar{u})\, d\bar{u}$$

If the mass is a one-dimensional rod that is nonuniform, then the heat capacity will vary as a function of x.

$$\Delta U = \int_0^u C(x, \bar{u})\, d\bar{u}$$

To obtain the change in thermal energy per unit mass, divide both sides by the mass m.

$$\frac{\Delta U}{m} = \int_0^u \frac{C(x,\bar{u})}{m}\,d\bar{u}$$

The heat capacity per unit mass is called the specific heat c—note that the adjective, "specific," implies per unit mass. Therefore,

$$\frac{\Delta U}{m} = \int_0^u c(x,\bar{u})\,d\bar{u}.$$

Part (b)

The law of conservation of energy states that energy is neither created nor destroyed. If some amount of thermal energy enters the left side of a shell at x, then that same amount must exit the right side of it at $x + \Delta x$ for the temperature to remain the same. If more (less) thermal energy enters at x than exits at $x + \Delta x$, then the amount of thermal energy in the shell will change, leading to an increase (decrease) in its temperature. The mathematical expression for this idea, an energy balance, is as follows.

rate of thermal energy in − rate of thermal energy out = rate of energy accumulation

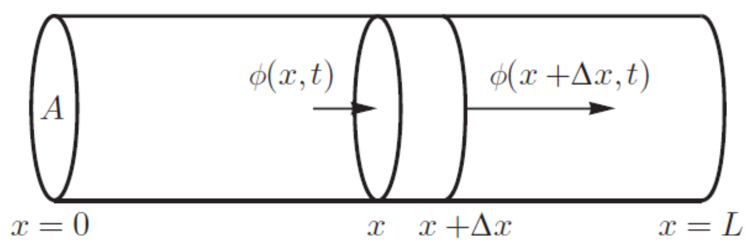

Figure 1: This is a schematic of the one-dimensional rod. The heat flow into the left side at x is the cross-sectional area there A times $\phi(x,t)$, and the heat flow out of the right side at $x + \Delta x$ is the cross-sectional area there A times $\phi(x + \Delta x, t)$.

Assuming there is a heat source per unit volume Q, then that will be included on the left in the terms for "rate of thermal energy in." The heat flux is defined to be the rate that thermal energy flows through the shell per unit area, and we denote it by $\phi = \phi(x,t)$. If we let U represent the amount of thermal energy of the rod, then the energy balance over it is

$$Q\Delta V + A\phi(x,t) - A\phi(x+\Delta x,t) = \left.\frac{dU}{dt}\right|_{\text{shell}}.$$

Factor $-A$ from the two terms containing ϕ.

$$Q\Delta V - A[\phi(x+\Delta x,t) - \phi(x,t)] = \left.\frac{dU}{dt}\right|_{\text{shell}} \tag{1}$$

We will show now that equation (1.2.3) in the text still holds. The thermal energy inside the shell is obtained by multiplying the thermal energy density $e(x,t)$ by the shell volume ΔV.

$$Q\Delta V - A[\phi(x+\Delta x,t) - \phi(x,t)] = \frac{d}{dt}e(x,t)\Delta V$$

The volume ΔV is the cross-sectional area A times thickness Δx.

$$QA\Delta x - A[\phi(x+\Delta x, t) - \phi(x,t)] = A\Delta x \frac{\partial e}{\partial t}$$

Divide both sides by $A\Delta x$.

$$Q - \frac{\phi(x+\Delta x, t) - \phi(x,t)}{\Delta x} = \frac{\partial e}{\partial t}$$

Now take the limit as $\Delta x \to 0$.

$$Q - \lim_{\Delta x \to 0} \frac{\phi(x+\Delta x, t) - \phi(x,t)}{\Delta x} = \frac{\partial e}{\partial t}$$

The limit is the first derivative of ϕ with respect to x. Therefore, equation (1.2.3) still holds.

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q} \tag{1.2.3}$$

The thermal energy per unit volume can be obtained by multiplying the thermal energy per unit mass in part (a) with the density $\rho(x)$.

$$\begin{aligned} e(x,t) &= \rho(x)\frac{\Delta U}{m} \\ &= \rho(x)\int_0^u c(x,\bar{u})\, d\bar{u} \end{aligned}$$

Substituting this into equation (1.2.3), we get

$$\frac{\partial}{\partial t}\left[\rho(x)\int_0^u c(x,\bar{u})\, d\bar{u}\right] = -\frac{\partial \phi}{\partial x} + Q(x,t),$$

or

$$\rho(x)\frac{\partial}{\partial t}\int_0^u c(x,\bar{u})\, d\bar{u} = -\frac{\partial \phi}{\partial x} + Q(x,t).$$

Apply the Leibnitz integration rule to differentiate the integral.

$$\rho(x)\left[\int_0^u \underbrace{\frac{\partial}{\partial t}c(x,\bar{u})}_{=0}\, d\bar{u} + c(x,u)\cdot\frac{\partial u}{\partial t} - c(x,0)\cdot 0\right] = -\frac{\partial \phi}{\partial x} + Q(x,t)$$

The equation simplifies to

$$\rho(x)c(x,u)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.$$

According to Fourier's law of heat conduction, the heat flux is proportional to the temperature gradient.

$$\phi = -K_0(x)\frac{\partial u}{\partial x},$$

where $K_0(x)$ is a proportionality constant known as the thermal conductivity. It varies as a function of x because the rod is nonuniform. As a result, the energy balance becomes an equation solely for the temperature.

$$\rho(x)c(x,u)\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[-K_0(x)\frac{\partial u}{\partial x}\right] + Q(x,t)$$

Therefore, the governing equation for the temperature in a nonuniform one-dimensional rod with heat source $Q(x,t)$ and specific heat varying as a function of temperature $c(x,u)$ is

$$\rho(x)c(x,u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + Q(x,t).$$