Exercise 1.3.3

Consider a bath containing a fluid of specific heat c_f and mass density ρ_f that surrounds the end $x = L$ of a one-dimensional rod. Suppose that the bath is rapidly stirred in a manner such that the bath temperature is approximately uniform throughout, equaling the temperature at $x = L$, $u(L, t)$. Assume that the bath is thermally insulated except at its perfect thermal contact with the rod, where the bath may be heated or cooled by the rod. Determine an equation for the temperature in the bath. (This will be a boundary condition at the end $x = L$.) (*Hint*: See Exercise 1.3.2.)

Solution

The rate that thermal energy flows from the end of the rod is given by the heat flux there $\phi(L, t)$ multiplied by the cross-sectional area there A. Because the lateral sides are insulated and energy is neither created nor destroyed, this thermal energy must flow into the bath. The mathematical expression for this idea, an energy balance, is given below.

$$\text{rate of thermal energy out of rod} = \text{rate of thermal energy into bath}$$

Using symbols, we have

$$A\phi(L,t) = \frac{dU}{dt},$$

where U represents the amount of thermal energy, and dU/dt represents how fast it changes in time. Since the bath is uniform in composition and temperature, the thermal energy in it is the product of the specific heat c_f, the mass m_f, and the temperature $u(L, t)$.

$$A\phi(L,t) = \frac{d}{dt}[m_f c_f u(L,t)]$$

The mass is the product of the mass density ρ_f and the volume V of the bath.

$$A\phi(L,t) = \frac{d}{dt}[V c_f \rho_f u(L,t)]$$

Bring the constants in front of the derivative.

$$A\phi(L,t) = V c_f \rho_f \frac{\partial u}{\partial t}(L,t)$$

The aim now is to write ϕ in terms of u by using Fourier's law of heat conduction, which states the heat flux is proportional to the temperature gradient.

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x},$$

where $K_0(x)$ is a proportionality constant known as the thermal conductivity. It varies as a function of x if the rod itself is assumed to be nonuniform in composition. Using this formula on the left side of the energy balance, we therefore have the boundary condition at $x = L$:

$$-AK_0(L)\frac{\partial u}{\partial x}(L,t) = V c_f \rho_f \frac{\partial u}{\partial t}(L,t).$$