Exercise 1.4.1

Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

(a)	$Q = 0,$	$u(0) = 0,$	$u(L) = T$
(b)	$Q = 0,$	$u(0) = T,$	$u(L) = 0$
(c)	$Q = 0,$	$\dfrac{\partial u}{\partial x}(0) = 0,$	$u(L) = T$
(d)	$Q = 0,$	$u(0) = T,$	$\dfrac{\partial u}{\partial x}(L) = \alpha$
(e)	$\dfrac{Q}{K_0} = 1,$	$u(0) = T_1,$	$u(L) = T_2$
(f)	$\dfrac{Q}{K_0} = x^2,$	$u(0) = T,$	$\dfrac{\partial u}{\partial x}(L) = 0$
(g)	$Q = 0,$	$u(0) = T,$	$\dfrac{\partial u}{\partial x}(L) + u(L) = 0$
(h)	$Q = 0,$	$\dfrac{\partial u}{\partial x}(0) - [u(0) - T] = 0,$	$\dfrac{\partial u}{\partial x}(L) = \alpha$

Solution

The heat equation for a one-dimensional rod with constant thermal properties, ρ, c, and K_0, and a heat source Q is

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q.$$

Part (a)

With $Q = 0$ the PDE reduces to

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} \quad \to \quad \frac{d^2 u}{dx^2} = 0$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice.

$$\frac{du}{dx} = C_1$$

$$u(x) = C_1 x + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$u(0) = C_2 = 0$$
$$u(L) = C_1 L + C_2 = T$$

Solving the second equation for C_1 gives $C_1 = T/L$. Therefore, the equilibrium temperature distribution is

$$u(x) = \frac{T}{L} x.$$

Part (b)

With $Q = 0$ the PDE reduces to

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice.

$$\frac{du}{dx} = C_1$$

$$u(x) = C_1 x + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$u(0) = C_2 = T$$
$$u(L) = C_1 L + C_2 = 0$$

Solving the second equation for C_1 gives $C_1 = -T/L$. Therefore, the equilibrium temperature distribution is

$$\begin{aligned} u(x) &= -\frac{T}{L}x + T \\ &= \frac{T}{L}(L - x). \end{aligned}$$

Part (c)

With $Q = 0$ the PDE reduces to

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice.

$$\frac{du}{dx} = C_1$$

Apply the first boundary condition here.

$$\frac{du}{dx}(0) = C_1 = 0$$

So we have

$$\frac{du}{dx} = 0.$$

Integrate both sides once more.

$$u(x) = C_2$$

Use the second boundary condition to determine C_2.

$$u(L) = C_2 = T$$

Therefore, the equilibrium temperature distribution is

$$u(x) = T.$$

Part (d)

With $Q = 0$ the PDE reduces to

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice.

$$\frac{du}{dx} = C_1$$

Apply the second boundary condition here.

$$\frac{du}{dx}(L) = C_1 = \alpha$$

So we have

$$\frac{du}{dx} = \alpha.$$

Integrate both sides once more.

$$u(x) = \alpha x + C_2$$

Use the first boundary condition to determine C_2.

$$u(0) = C_2 = T$$

Therefore, the equilibrium temperature distribution is

$$u(x) = \alpha x + T.$$

Part (e)

With $Q = K_0$ the PDE reduces to

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + K_0.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} + K_0 \quad \rightarrow \quad \frac{d^2 u}{dx^2} = -1$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice.

$$\frac{du}{dx} = -x + C_1$$

$$u(x) = -\frac{x^2}{2} + C_1x + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$u(0) = C_2 = T_1$$
$$u(L) = -\frac{L^2}{2} + C_1L + C_2 = T_2$$

Solving the second equation for C_1 gives

$$C_1 = \frac{T_2 - T_1}{L} + \frac{L}{2}.$$

Therefore, the equilibrium temperature distribution is

$$u(x) = -\frac{x^2}{2} + \left(\frac{T_2 - T_1}{L} + \frac{L}{2}\right) x + T_1.$$

Part (f)

With $Q = K_0x^2$ the PDE reduces to

$$\rho c\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + K_0x^2.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0\frac{d^2u}{dx^2} + K_0x^2 \quad \rightarrow \quad \frac{d^2u}{dx^2} = -x^2$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice.

$$\frac{du}{dx} = -\frac{x^3}{3} + C_1$$

Apply the second boundary condition here.

$$\frac{du}{dx}(L) = -\frac{L^3}{3} + C_1 = 0 \quad \rightarrow \quad C_1 = \frac{L^3}{3}$$

So we have

$$\frac{du}{dx} = -\frac{x^3}{3} + \frac{L^3}{3}.$$

Integrate both sides once more.

$$u(x) = -\frac{x^4}{12} + \frac{L^3}{3}x + C_2$$

Use the first boundary condition to determine C_2.

$$u(0) = C_2 = T$$

Therefore, the equilibrium temperature distribution is

$$u(x) = -\frac{x^4}{12} + \frac{L^3}{3}x + T.$$

Part (g)

With $Q = 0$ the PDE reduces to

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice.

$$\frac{du}{dx} = C_1$$

$$u(x) = C_1 x + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$u(0) = C_2 = T$$

$$\frac{du}{dx}(L) + u(L) = C_1 + C_1 L + C_2 = 0$$

Solving the second equation for C_1 gives

$$C_1 = -\frac{T}{1+L}.$$

Therefore, the equilibrium temperature distribution is

$$\begin{aligned} u(x) &= -\frac{T}{1+L}x + T \\ &= \frac{T}{L+1}(L + 1 - x). \end{aligned}$$

Part (h)

With $Q = 0$ the PDE reduces to

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0$$

The general solution to this ODE is obtained by integrating both sides with respect to x twice.

$$\frac{du}{dx} = C_1$$

Apply the second boundary condition here.

$$\frac{du}{dx}(L) = C_1 = \alpha$$

So we have

$$\frac{du}{dx} = \alpha.$$

Integrate both sides once more.

$$u(x) = \alpha x + C_2$$

Use the first boundary condition to determine C_2.

$$\frac{du}{dx}(0) - [u(0) - T] = \alpha - [C_2 - T] = 0$$

Solving the equation gives $C_2 = \alpha + T$. Therefore, the equilibrium temperature distribution is

$$\begin{aligned} u(x) &= \alpha x + \alpha + T \\ &= \alpha(x + 1) + T. \end{aligned}$$