Exercise 1.4.7

For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

$$\textbf{(a)}\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1, \qquad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = 1, \quad \frac{\partial u}{\partial x}(L,t) = \beta$$

$$\textbf{(b)}\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = 1, \quad \frac{\partial u}{\partial x}(L,t) = \beta$$

$$\textbf{(c)}\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

Solution

The rod in (a) has constant physical properties and a constant heat source $Q = 1$. The heat flow is specified at its ends, and it has an initial temperature distribution $u(x, 0) = f(x)$. The rod in (b) has constant physical properties and no heat source. The heat flow is specified at its ends, and it has an initial temperature distribution $u(x, 0) = f(x)$. The rod in (c) has constant physical properties and a steady heat source $Q(x) = x - \beta$. The ends are insulated, and it has an initial temperature distribution $u(x, 0) = f(x)$.

Part (a)

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = \frac{d^2u}{dx^2} + 1 \quad \rightarrow \quad \frac{d^2u}{dx^2} = -1$$

This differential equation can be solved by integrating both sides with respect to x twice. After the first integration, we get

$$\frac{du}{dx} = -x + C_1.$$

Apply the boundary conditions at $x = 0$ and $x = L$ to determine C_1 and β.

$$\frac{du}{dx}(0) = C_1 = 1$$
$$\frac{du}{dx}(L) = -L + C_1 = \beta$$

In order for there to be an equilibrium temperature distribution, β must be equal to $1 - L$.

$$\frac{du}{dx} = -x + 1$$

Integrate both sides with respect to x once more.

$$u(x) = -\frac{x^2}{2} + x + C_2$$

The final constant can be found by integrating both sides of the PDE over the rod's length from 0 to L.

$$\int_0^L \frac{\partial u}{\partial t}\, dx = \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + 1\right) dx$$

Bring the time derivative in front of the integral on the left side. It becomes a total derivative because the definite integral wipes out the x variable. Split up the integral on the right side into two and evaluate them.

$$\begin{aligned}
\frac{d}{dt}\int_0^L u(x,t)\,dx &= \int_0^L \frac{\partial^2 u}{\partial x^2}\,dx + \int_0^L dx \\
&= \left.\frac{\partial u}{\partial x}\right|_0^L + L \\
&= \underbrace{\frac{\partial u}{\partial x}(L,t)}_{=\,\beta} - \underbrace{\frac{\partial u}{\partial x}(0,t)}_{=\,1} + L \\
&= \beta - 1 + L \\
&= 0
\end{aligned}$$

Integrate both sides with respect to t.

$$\int_0^L u(x,t)\,dx = \text{constant}$$

As a result, the integral of u over the rod's length is the same at any time, including at equilibrium.

$$\int_0^L u(x,0)\,dx = \int_0^L u(x,\infty)\,dx = \text{constant}$$

Substitute the prescribed initial condition into the integrand on the left side and the equilibrium temperature distribution into the right side.

$$\int_0^L f(x)\,dx = \int_0^L \left(-\frac{x^2}{2} + x + C_2\right) dx$$

We now have an equation for C_2. Proceed to evaluate the integral and solve for it.

$$\int_0^L f(x)\,dx = -\frac{L^3}{6} + \frac{L^2}{2} + C_2 L$$

So then

$$\begin{aligned}
C_2 &= \frac{1}{L}\left[\frac{L^3}{6} - \frac{L^2}{2} + \int_0^L f(x)\,dx\right] \\
&= \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx.
\end{aligned}$$

Therefore, assuming $\beta = 1 - L$, the equilibrium temperature distribution is

$$u(x) = -\frac{x^2}{2} + x + \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx.$$

Part (b)

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = \frac{d^2u}{dx^2}$$

This differential equation can be solved by integrating both sides with respect to x twice. After the first integration, we get

$$\frac{du}{dx} = C_3.$$

Apply the boundary conditions at $x = 0$ and $x = L$ to determine C_3 and β.

$$\frac{du}{dx}(0) = C_3 = 1$$
$$\frac{du}{dx}(L) = C_3 = \beta$$

In order for there to be an equilibrium temperature distribution, β must be equal to 1.

$$\frac{du}{dx} = 1$$

Integrate both sides with respect to x once more.

$$u(x) = x + C_4$$

The final constant can be found by integrating both sides of the PDE over the rod's length from 0 to L.

$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2}\,dx$$

Bring the time derivative in front of the integral on the left side. It becomes a total derivative because the definite integral wipes out the x variable. Evaluate the right side.

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)\,dx &= \left.\frac{\partial u}{\partial x}\right|_0^L \\ &= \underbrace{\frac{\partial u}{\partial x}(L,t)}_{=\,\beta} - \underbrace{\frac{\partial u}{\partial x}(0,t)}_{=\,1} \\ &= \beta - 1 \\ &= 0\end{aligned}$$

Integrate both sides with respect to t.

$$\int_0^L u(x,t)\,dx = \text{constant}$$

As a result, the integral of u over the rod's length is the same at any time, including at equilibrium.

$$\int_0^L u(x,0)\,dx = \int_0^L u(x,\infty)\,dx = \text{constant}$$

Substitute the prescribed initial condition into the integrand on the left side and the equilibrium temperature distribution into the right side.

$$\int_0^L f(x)\,dx = \int_0^L (x + C_4)dx$$

We now have an equation for C_4. Proceed to evaluate the integral and solve for it.

$$\int_0^L f(x)\,dx = \frac{L^2}{2} + C_4 L$$

So then

$$\begin{aligned} C_4 &= \frac{1}{L}\left[-\frac{L^2}{2} + \int_0^L f(x)\,dx\right] \\ &= -\frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx. \end{aligned}$$

Therefore, assuming $\beta = 1$, the equilibrium temperature distribution is

$$u(x) = x - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx.$$

Part (c)

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = \frac{d^2u}{dx^2} + x - \beta \quad \rightarrow \quad \frac{d^2u}{dx^2} = \beta - x$$

This differential equation can be solved by integrating both sides with respect to x twice. After the first integration, we get

$$\frac{du}{dx} = \beta x - \frac{x^2}{2} + C_5.$$

Apply the boundary conditions at $x = 0$ and $x = L$ to determine C_5 and β.

$$\begin{aligned} &\frac{du}{dx}(0) = C_5 = 0 \\ &\frac{du}{dx}(L) = \beta L - \frac{L^2}{2} + C_5 = 0 \quad \rightarrow \quad \beta = \frac{L}{2} \end{aligned}$$

In order for there to be an equilibrium temperature distribution, β must be equal to $L/2$.

$$\frac{du}{dx} = \frac{L}{2}x - \frac{x^2}{2}$$

Integrate both sides with respect to x once more.

$$u(x) = \frac{L}{4}x^2 - \frac{x^3}{6} + C_6$$

The final constant can be found by integrating both sides of the PDE over the rod's length from 0 to L.

$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + x - \beta\right)dx$$

Bring the time derivative in front of the integral on the left side. It becomes a total derivative because the definite integral wipes out the x variable. Split up the integral on the right side into three and evaluate them.

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)\,dx &= \int_0^L \frac{\partial^2 u}{\partial x^2}\,dx + \int_0^L x\,dx - \beta\int_0^L dx \\ &= \frac{\partial u}{\partial x}\bigg|_0^L + \frac{L^2}{2} - \beta L \\ &= \underbrace{\frac{\partial u}{\partial x}(L,t)}_{=\,0} - \underbrace{\frac{\partial u}{\partial x}(0,t)}_{=\,0} + \frac{L^2}{2} - \beta L \\ &= \frac{L^2}{2} - \beta L \\ &= 0\end{aligned}$$

Integrate both sides with respect to t.

$$\int_0^L u(x,t)\,dx = \text{constant}$$

As a result, the integral of u over the rod's length is the same at any time, including at equilibrium.

$$\int_0^L u(x,0)\,dx = \int_0^L u(x,\infty)\,dx = \text{constant}$$

Substitute the prescribed initial condition into the integrand on the left side and the equilibrium temperature distribution into the right side.

$$\int_0^L f(x)\,dx = \int_0^L \left(\frac{L}{4}x^2 - \frac{x^3}{6} + C_6\right)dx$$

We now have an equation for C_6. Proceed to evaluate the integral and solve for it.

$$\int_0^L f(x)\,dx = \frac{L^4}{12} - \frac{L^4}{24} + C_6 L$$

So then

$$\begin{aligned}C_6 &= \frac{1}{L}\left[-\frac{L^4}{24} + \int_0^L f(x)\,dx\right] \\ &= -\frac{L^3}{24} + \frac{1}{L}\int_0^L f(x)\,dx.\end{aligned}$$

Therefore, assuming $\beta = L/2$, the equilibrium temperature distribution is

$$u(x) = \frac{L}{4}x^2 - \frac{x^3}{6} - \frac{L^3}{24} + \frac{1}{L}\int_0^L f(x)\,dx.$$