Exercise 1.5.10

Determine the equilibrium temperature distribution inside a circle ($r \leq r_0$) if the boundary is fixed at temperature T_0.

Solution

The governing equation for the temperature in a circle, assuming radial symmetry, is

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of r now.

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) \quad \to \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$$

To solve the differential equation, multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{du}{dr} = C_1$$

Divide both sides by r.

$$\frac{du}{dr} = \frac{C_1}{r}$$

Integrate both sides with respect to r once more.

$$u(r) = C_1 \ln r + C_2$$

In order for the temperature of the circle to remain finite as $r \to 0$, we require that $C_1 = 0$.

$$u(r) = C_2$$

Apply the boundary condition here to determine C_2: At $r = r_0$ the temperature is T_0.

$$u(r_0) = C_2 = T_0$$

Therefore, the equilibrium temperature distribution is unsurprisingly

$$u(r) = T_0.$$