Exercise 1.5.17

Derive the integral conservation law for any three dimensional object (with constant thermal properties) by integrating the heat equation (1.5.11) (assuming no sources). Show that the result is equivalent to (1.5.1).

Solution

Equation (1.5.11) in the textbook is the heat equation in three dimensions without a source.

$$\frac{\partial u}{\partial t} = k\nabla^2 u \tag{1.5.11}$$

Replace k with $K_0/\rho c$ and multiply both sides by ρc.

$$\rho c\frac{\partial u}{\partial t} = K_0\nabla^2 u$$

Bring ρ and c inside the time derivative.

$$\frac{\partial(\rho c u)}{\partial t} = K_0\nabla^2 u$$

The product of mass density, specific heat, and temperature is the thermal energy density e.

$$\frac{\partial e}{\partial t} = K_0\nabla^2 u$$

Integrate both sides over the volume V of the object in order to get the total thermal energy in it, $\int_V e\,dV$.

$$\iiint_V \frac{\partial e}{\partial t}\,dV = \iiint_V K_0\nabla^2 u\,dV$$

Bring the time derivative in front of the volume integral.

$$\frac{d}{dt}\iiint_V e\,dV = \iiint_V K_0\nabla^2 u\,dV$$

The definite volume integral wipes out the spatial variables in e, so the time derivative is a total derivative in front of the integral.

$$\frac{d}{dt}\iiint_V e\,dV = K_0\iiint_V \nabla\cdot\nabla u\,dV$$

Apply the divergence theorem here to change the volume integral on the right side into an integral over the object's surface S. Let $\mathbf{n}$ be a unit vector perpendicular to S.

$$\begin{aligned}\frac{d}{dt}\iiint_V e\,dV &= K_0\oiint_S \nabla u\cdot\mathbf{n}\,dS \\ &= -\oiint_S(-K_0\nabla u\cdot\mathbf{n})\,dS\end{aligned}$$

According to Fourier's law of conduction, the heat flux is proportional to the temperature gradient.

$$\phi = -K_0\nabla u,$$

where K_0 is a proportionality constant known as the thermal conductivity. Substitute this result into the surface integral to obtain equation (1.5.1) in the textbook.

$$\frac{d}{dt}\iiint_V e\,dV = -\oiint_S \phi\cdot\mathbf{n}\,dS \tag{1.5.1}$$

This equation tells us that the rate the thermal energy in the object changes is equal to the net rate that heat flows through the boundary of the object.

rate of energy accumulation = rate of energy in − rate of energy out.

The significance of the minus sign in front of the surface integral is that thermal energy is assumed to flow in the direction of increasing $\mathbf{x}$, that is, out of the object.