Exercise 1.5.25

Suppose a sphere of radius 2 satisfies $\frac{\partial u}{\partial t} = \nabla^2 u + 5$ with $u(x, y, z, 0) = f(x, y, z)$ and on the surface of the sphere it is given that $\nabla u \cdot \hat{n} = 6$, where $\hat{n}$ is a unit outward normal vector. Calculate the total thermal energy for this sphere as a function of time. (*Hint*: Use the divergence theorem.)

Solution

The governing equation for the sphere's temperature u is

$$\frac{\partial u}{\partial t} = \nabla^2 u + 5.$$

Comparing this to the general form of the heat equation, we see that the mass density ρ and specific heat c are equal to 1 and that the heat source is $Q = 5$. The thermal energy density e is $\rho c u = u$, so the left side can be written in terms of e.

$$\frac{\partial e}{\partial t} = \nabla^2 u + 5$$

To obtain the total thermal energy in the sphere, integrate both sides over the sphere's volume V.

$$\iiint_V \frac{\partial e}{\partial t}\, dV = \iiint_V (\nabla^2 u + 5)\, dV$$

Bring the time derivative in front of the volume integral on the left and split the volume integral on the right into two.

$$\frac{d}{dt}\iiint_V e\, dV = \iiint_V \nabla^2 u\, dV + 5\iiint_V dV$$

The triple integral on the left represents the total thermal energy in the sphere. The second integral on the right side is the sphere's volume.

$$= \iiint_V \nabla \cdot \nabla u\, dV + 5 \cdot \frac{4}{3}\pi(2)^3$$

Apply the divergence theorem here to the remaining triple integral. The volume integral becomes an integral over the sphere's surface.

$$= \oiint_S \nabla u \cdot \hat{n}\, dS + \frac{160\pi}{3}$$

Use the fact that $\nabla u \cdot \hat{n} = 6$.

$$= \oiint_S 6\, dS + \frac{160\pi}{3}$$

The closed surface integral is just 6 times the surface area of the sphere.

$$= 6 \cdot 4\pi(2)^2 + \frac{160\pi}{3}$$

$$= 96\pi + \frac{160\pi}{3}$$

$$\frac{d}{dt}\iiint_V e\, dV = \frac{448\pi}{3}$$

To solve for the total thermal energy, integrate both sides with respect to t.

$$\iiint_V e\,dV = \frac{448\pi}{3}t + C$$

To determine C, set $t = 0$ and use the prescribed initial condition $u(x, y, z, 0) = f(x, y, z)$.

$$C = \iiint_V e(x, y, z, 0)\,dV = \iiint_V u(x, y, z, 0)\,dV = \iiint_V f(x, y, z)\,dV = \iiint_V f(x, y, z)\,dx\,dy\,dz$$

Therefore, the total thermal energy in the sphere is

$$\iiint_V e\,dV = \frac{448\pi}{3}t + \iiint_V f(x, y, z)\,dx\,dy\,dz.$$