Exercise 1.2.8

If \( u(x,t) \) is known, give an expression for the total thermal energy contained in a rod \((0 < x < L)\).

Solution

The thermal energy in the rod is equal to the mass \( m \) times specific heat \( c \) times temperature \( u \). For a nonuniform rod, the total thermal energy is obtained by integrating over the rod’s mass.

\[
\text{Total Thermal Energy} = \int_{\text{rod}} cu \, dm
\]

The specific heat is assumed to vary as a function of \( x \), and \( u = u(x,t) \). The mass is density times volume, so the differential is \( dm = \rho(x) \, dV \). The integral changes into one over the rod’s volume.

\[
\text{Total Thermal Energy} = \int_{\text{rod}} \rho(x)c(x)u(x,t) \, dV
\]

Assuming that the cross-sectional area of the rod also varies as a function of \( x \), the differential of volume is \( dV = A(x) \, dx \). The integral changes into one over the rod’s length.

\[
\text{Total Thermal Energy} = \int_{\text{rod}} \rho(x)c(x)u(x,t)A(x) \, dx
\]

As the rod goes from \( 0 < x < L \), the total energy is therefore

\[
\text{Total Thermal Energy} = \int_{0}^{L} \rho(x)c(x)u(x,t)A(x) \, dx.
\]