Exercise 1.4.1

Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

(a) \( Q = 0, \quad u(0) = 0, \quad u(L) = T \)

(b) \( Q = 0, \quad u(0) = T, \quad u(L) = 0 \)

(c) \( Q = 0, \quad \frac{\partial u}{\partial x}(0) = 0, \quad u(L) = T \)

(d) \( Q = 0, \quad u(0) = T, \quad \frac{\partial u}{\partial x}(L) = \alpha \)

(e) \( \frac{Q}{K_0} = 1, \quad u(0) = T_1, \quad u(L) = T_2 \)

(f) \( \frac{Q}{K_0} = x^2, \quad u(0) = T, \quad \frac{\partial u}{\partial x}(L) = 0 \)

(g) \( Q = 0, \quad u(0) = T, \quad \frac{\partial u}{\partial x}(L) + u(L) = 0 \)

(h) \( Q = 0, \quad \frac{\partial u}{\partial x}(0) - [u(0) - T] = 0, \quad \frac{\partial u}{\partial x}(L) = \alpha \)

Solution

The heat equation for a one-dimensional rod with constant thermal properties, \( \rho, c, \) and \( K_0, \) and a heat source \( Q \) is

\[
\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q.
\]

Part (a)

With \( Q = 0 \) the PDE reduces to

\[
\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}.
\]

At equilibrium the temperature does not change in time, so \( \partial u/\partial t \) vanishes. \( u \) is only a function of \( x \) now.

\[0 = K_0 \frac{d^2 u}{dx^2} \rightarrow \frac{d^2 u}{dx^2} = 0\]

The general solution to this ODE is obtained by integrating both sides with respect to \( x \) twice.

\[\frac{du}{dx} = C_1\]

\[u(x) = C_1 x + C_2\]

Apply the boundary conditions here to determine \( C_1 \) and \( C_2.\)

\[u(0) = C_2 = 0\]

\[u(L) = C_1 L + C_2 = T\]

Solving the second equation for \( C_1 \) gives \( C_1 = T/L. \) Therefore, the equilibrium temperature distribution is

\[u(x) = \frac{T}{L} x.\]
Part (b)

With $Q = 0$ the PDE reduces to
\[ \rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}. \]

At equilibrium the temperature does not change in time, so $\partial u / \partial t$ vanishes. $u$ is only a function of $x$ now.

\[ 0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0 \]

The general solution to this ODE is obtained by integrating both sides with respect to $x$ twice.

\[ \frac{du}{dx} = C_1 \]

\[ u(x) = C_1 x + C_2 \]

Apply the boundary conditions here to determine $C_1$ and $C_2$.

\[ u(0) = C_2 = T \]
\[ u(L) = C_1 L + C_2 = 0 \]

Solving the second equation for $C_1$ gives $C_1 = -T/L$. Therefore, the equilibrium temperature distribution is

\[ u(x) = -\frac{T}{L} x + T \]
\[ = \frac{T}{L} (L - x). \]

Part (c)

With $Q = 0$ the PDE reduces to
\[ \rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}. \]

At equilibrium the temperature does not change in time, so $\partial u / \partial t$ vanishes. $u$ is only a function of $x$ now.

\[ 0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0 \]

The general solution to this ODE is obtained by integrating both sides with respect to $x$ twice.

\[ \frac{du}{dx} = C_1 \]

Apply the first boundary condition here.

\[ \frac{du}{dx}(0) = C_1 = 0 \]

So we have

\[ \frac{du}{dx} = 0. \]

Integrate both sides once more.

\[ u(x) = C_2 \]
Use the second boundary condition to determine $C_2$.

$$u(L) = C_2 = T$$

Therefore, the equilibrium temperature distribution is

$$u(x) = T.$$  

**Part (d)**

With $Q = 0$ the PDE reduces to

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}.$$  

At equilibrium the temperature does not change in time, so $\partial u / \partial t$ vanishes. $u$ is only a function of $x$ now.

$$0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0$$

The general solution to this ODE is obtained by integrating both sides with respect to $x$ twice.

$$\frac{du}{dx} = C_1$$

Apply the second boundary condition here.

$$\frac{du}{dx}(L) = C_1 = \alpha$$

So we have

$$\frac{du}{dx} = \alpha.$$  

Integrate both sides once more.

$$u(x) = \alpha x + C_2$$

Use the first boundary condition to determine $C_2$.

$$u(0) = C_2 = T$$

Therefore, the equilibrium temperature distribution is

$$u(x) = \alpha x + T.$$  

**Part (e)**

With $Q = K_0$ the PDE reduces to

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + K_0.$$  

At equilibrium the temperature does not change in time, so $\partial u / \partial t$ vanishes. $u$ is only a function of $x$ now.

$$0 = K_0 \frac{d^2 u}{dx^2} + K_0 \quad \rightarrow \quad \frac{d^2 u}{dx^2} = -1$$

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The general solution to this ODE is obtained by integrating both sides with respect to $x$ twice.

\[
\frac{du}{dx} = -x + C_1
\]

\[
u(x) = -\frac{x^2}{2} + C_1 x + C_2
\]

Apply the boundary conditions here to determine $C_1$ and $C_2$.

\[
u(0) = C_2 = T_1
\]

\[
u(L) = -\frac{L^2}{2} + C_1 L + C_2 = T_2
\]

Solving the second equation for $C_1$ gives

\[
C_1 = \frac{T_2 - T_1}{L} + \frac{L}{2}.
\]

Therefore, the equilibrium temperature distribution is

\[
u(x) = -\frac{x^2}{2} + \left(\frac{T_2 - T_1}{L} + \frac{L}{2}\right) x + T_1.
\]

**Part (f)**

With $Q = K_0 x^2$ the PDE reduces to

\[
\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + K_0 x^2.
\]

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. $u$ is only a function of $x$ now.

\[
0 = K_0 \frac{d^2 u}{dx^2} + K_0 x^2 \quad \rightarrow \quad \frac{d^2 u}{dx^2} = -x^2
\]

The general solution to this ODE is obtained by integrating both sides with respect to $x$ twice.

\[
\frac{du}{dx} = -\frac{x^3}{3} + C_1
\]

Apply the second boundary condition here.

\[
\frac{du}{dx}(L) = -\frac{L^3}{3} + C_1 = 0 \quad \rightarrow \quad C_1 = \frac{L^3}{3}
\]

So we have

\[
\frac{du}{dx} = -\frac{x^3}{3} + \frac{L^3}{3}.
\]

Integrate both sides once more.

\[
u(x) = -\frac{x^4}{12} + \frac{L^3}{3} x + C_2
\]

Use the first boundary condition to determine $C_2$.

\[
u(0) = C_2 = T
\]
Therefore, the equilibrium temperature distribution is

\[ u(x) = -\frac{x^4}{12} + \frac{L^3}{3}x + T. \]

**Part (g)**

With \( Q = 0 \) the PDE reduces to

\[ \rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}. \]

At equilibrium the temperature does not change in time, so \( \partial u/\partial t \) vanishes. \( u \) is only a function of \( x \) now.

\[ 0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0 \]

The general solution to this ODE is obtained by integrating both sides with respect to \( x \) twice.

\[ du = C_1 \]

\[ u(x) = C_1 x + C_2 \]

Apply the boundary conditions here to determine \( C_1 \) and \( C_2 \).

\[ u(0) = C_2 = T \]

\[ \frac{du}{dx}(L) + u(L) = C_1 + C_1 L + C_2 = 0 \]

Solving the second equation for \( C_1 \) gives

\[ C_1 = -\frac{T}{1+L}. \]

Therefore, the equilibrium temperature distribution is

\[ u(x) = -\frac{T}{1+L} x + T \]

\[ = \frac{T}{L+1}(L+1-x). \]

**Part (h)**

With \( Q = 0 \) the PDE reduces to

\[ \rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}. \]

At equilibrium the temperature does not change in time, so \( \partial u/\partial t \) vanishes. \( u \) is only a function of \( x \) now.

\[ 0 = K_0 \frac{d^2 u}{dx^2} \quad \rightarrow \quad \frac{d^2 u}{dx^2} = 0 \]

The general solution to this ODE is obtained by integrating both sides with respect to \( x \) twice.

\[ du = C_1 \]

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Apply the second boundary condition here.

$$\frac{du}{dx}(L) = C_1 = \alpha$$

So we have

$$\frac{du}{dx} = \alpha.$$ 

Integrate both sides once more.

$$u(x) = \alpha x + C_2$$

Use the first boundary condition to determine \(C_2\).

$$\frac{du}{dx}(0) - [u(0) - T] = \alpha - [C_2 - T] = 0$$

Solving the equation gives \(C_2 = \alpha + T\). Therefore, the equilibrium temperature distribution is

$$u(x) = \alpha x + \alpha + T$$

$$= \alpha(x + 1) + T.$$