Exercise 1.4.10

Suppose \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4 \), \( u(x,0) = f(x) \), \( \frac{\partial u}{\partial x}(0,t) = 5 \), \( \frac{\partial u}{\partial x}(L,t) = 6 \). Calculate the total thermal energy in the one-dimensional rod (as a function of time).

Solution

The governing equation for the rod’s temperature \( u \) is

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4.
\]

Comparing this to the general form of the heat equation, we see that the mass density \( \rho \) and specific heat \( c \) are equal to 1 and that the heat source is \( Q = 4 \). The thermal energy density \( e \) is \( \rho cu = u \), so the left side can be written in terms of \( e \).

\[
\frac{\partial e}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4
\]

To obtain the total thermal energy in the rod, integrate both sides over the rod’s volume \( V \).

\[
\int_V \frac{\partial e}{\partial t} \, dV = \int_V \left( \frac{\partial^2 u}{\partial x^2} + 4 \right) \, dV
\]

Bring the time derivative in front of the volume integral on the left.

\[
dt \int_V e \, dV = \int_V \left( \frac{\partial^2 u}{\partial x^2} + 4 \right) \, dV
\]

The volume integral on the left represents the total thermal energy in the rod, and that’s what we intend to solve for. The rod has a constant cross-sectional area \( A \), so the volume differential is \( dV = A \, dx \). The volume integral on the right side will be replaced by one over the rod’s length.

\[
\int_0^L \frac{\partial e}{\partial t} \, dx = \int_0^L \left( \frac{\partial^2 u}{\partial x^2} + 4 \right) \, dx
\]

Integrate both sides with respect to \( t \).

\[
\int_V e \, dV = A(1 + 4L)t + U_0
\]

The constant of integration \( U_0 \) is the initial thermal energy in the rod. In order to determine it, we will make use of the initial condition \( u(x,0) = f(x) \). Change \( e \) back in terms of \( u \) and write \( dV = A \, dx \).

\[
\int_0^L u(x,t)A \, dx = A(1 + 4L)t + U_0
\]
Bring $A$ in front of the integral and set $t = 0$ in the equation.

$$A \int_0^L u(x, 0) \, dx = U_0$$

Use the initial condition.

$$A \int_0^L f(x) \, dx = U_0$$

Therefore, the thermal energy in the rod as a function of time is

$$\int_V e \, dV = A(1 + 4L)t + A \int_0^L f(x) \, dx.$$