Exercise 1.4.11

Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$, $u(x, 0) = f(x)$, $\frac{\partial u}{\partial x}(0, t) = \beta$, $\frac{\partial u}{\partial x}(L, t) = 7$.

(a) Calculate the total thermal energy in the one-dimensional rod (as a function of time).

(b) From part (a), determine a value of $\beta$ for which an equilibrium exists. For this value of $\beta$, determine $\lim_{t \to \infty} u(x, t)$.

Part (a)

The governing equation for the rod’s temperature $u$ is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x.$$ 

Comparing this to the general form of the heat equation, we see that the mass density $\rho$ and specific heat $c$ are equal to 1 and that the heat source is $Q = x$. The thermal energy density $e$ is $\rho c u = u$, so the left side can be written in terms of $e$.

$$\frac{\partial e}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$$

To obtain the total thermal energy in the rod, integrate both sides over the rod’s volume $V$.

$$\int_V \frac{\partial e}{\partial t} \, dV = \int_V \left( \frac{\partial^2 u}{\partial x^2} + x \right) \, dV$$

Bring the time derivative in front of the volume integral on the left.

$$\frac{d}{dt} \int_V e \, dV = \int_V \left( \frac{\partial^2 u}{\partial x^2} + x \right) \, dV$$

The volume integral on the left represents the total thermal energy in the rod, and that’s what we intend to solve for. The rod has a constant cross-sectional area $A$, so the volume differential is $dV = A \, dx$. The volume integral on the right side will be replaced by one over the rod’s length.

$$\frac{d}{dt} \int_V e \, dV = \int_0^L \left( \frac{\partial^2 u}{\partial x^2} + x \right) A \, dx$$

Integrate both sides with respect to $t$.

$$\int_V e \, dV = A \left( 7 - \beta + \frac{L^2}{2} \right) t + U_0$$
The constant of integration $U_0$ is the initial thermal energy in the rod. In order to determine it, we will make use of the initial condition $u(x,0) = f(x)$. Change $e$ back in terms of $u$ and write $dV = A\, dx$.

$$\int_0^L u(x,t)A\, dx = A \left( 7 - \beta + \frac{L^2}{2} \right) t + U_0$$

Bring $A$ in front of the integral and set $t = 0$ in the equation.

$$A \int_0^L u(x,0)\, dx = U_0$$

Use the initial condition.

$$A \int_0^L f(x)\, dx = U_0$$

Therefore, the thermal energy in the rod as a function of time is

$$\int_V e\, dV = A \left( 7 - \beta + \frac{L^2}{2} \right) t + A \int_0^L f(x)\, dx.$$ 

**Part (b)**

Equilibrium can only occur if the thermal energy in the rod is constant. This happens if

$$7 - \beta + \frac{L^2}{2} = 0 \quad \Rightarrow \quad \beta = 7 + \frac{L^2}{2}.$$ 

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. $u$ is only a function of $x$ now.

$$0 = \frac{d^2 u}{dx^2} + x \quad \Rightarrow \quad \frac{d^2 u}{dx^2} = -x$$

This differential equation can be solved by integrating both sides with respect to $x$ twice. After the first integration, we get

$$\frac{du}{dx} = -\frac{x^2}{2} + C_1.$$ 

Apply the boundary conditions here to determine $C_1$.

$$\frac{du}{dx}(0) = C_1 = \beta$$

$$\frac{du}{dx}(L) = -\frac{L^2}{2} + C_1 = 7 \quad \Rightarrow \quad C_1 = 7 + \frac{L^2}{2}$$

So then

$$\frac{du}{dx} = -\frac{x^2}{2} + 7 + \frac{L^2}{2}.$$ 

Integrate both sides with respect to $x$ a second time.

$$u(x) = -\frac{x^3}{6} + \left( 7 + \frac{L^2}{2} \right) x + C_2$$

The result from part (a) will be used to determine $C_2$. If $\beta = 7 + L^2/2$, then it simplifies to

$$\int_V e\, dV = A \int_0^L f(x)\, dx.$$ 

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Change $e$ back to $u$ and $dV$ to $Adx$.

$$\int_0^L u(x,t)A\, dx = A\int_0^L f(x)\, dx$$

Divide both sides by $A$ and then set $t = \infty$.

$$\int_0^L u(x,\infty)\, dx = \int_0^L f(x)\, dx$$

Substitute the equilibrium temperature for $u(x, \infty)$.

$$\int_0^L \left[ -\frac{x^3}{6} + \left(7 + \frac{L^2}{2}\right)x + C_2\right] dx = \int_0^L f(x)\, dx$$

We now have an equation for $C_2$. Evaluate the integral on the left side.

$$-\frac{L^4}{24} + \left(7 + \frac{L^2}{2}\right)\frac{L^2}{2} + C_2L = \int_0^L f(x)\, dx$$

Simplify the left side.

$$\frac{5L^4}{24} + \frac{7L^2}{2} + C_2L = \int_0^L f(x)\, dx$$

So we have

$$C_2 = -\frac{5L^3}{24} - \frac{7L}{2} + \frac{1}{L} \int_0^L f(x)\, dx.$$ 

Therefore, assuming $\beta = 7 + L^2/2$, the equilibrium temperature distribution is

$$u(x) = -\frac{x^3}{6} + \left(7 + \frac{L^2}{2}\right)x - \frac{5L^3}{24} - \frac{7L}{2} + \frac{1}{L} \int_0^L f(x)\, dx.$$