Exercise 1.4.6

The two ends of a uniform rod of length L are insulated. There is a constant source of thermal energy $Q_0 \neq 0$, and the temperature is initially $u(x, 0) = f(x)$.

(a) Show mathematically that there does not exist any equilibrium temperature distribution. Briefly explain physically.

(b) Calculate the total thermal energy in the entire rod.

Solution

The governing equation for the temperature in the rod, assuming it has constant physical properties, is

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q_0.$$

Part (a)

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of x now.

$$0 = K_0 \frac{d^2 u}{dx^2} + Q_0 \quad \rightarrow \quad \frac{d^2 u}{dx^2} = -\frac{Q_0}{K_0}$$

This differential equation can be solved by integrating both sides with respect to x twice. After the first integration, we get

$$\frac{du}{dx} = -\frac{Q_0}{K_0} x + C_1.$$

If the rod is insulated at both ends, then the boundary conditions are $du/dx(0) = du/dx(L) = 0$.

$$\begin{aligned} &\frac{du}{dx}(0) = C_1 = 0 \\ &\frac{du}{dx}(L) = -\frac{Q_0}{K_0} L + C_1 = 0 \quad \rightarrow \quad C_1 = \frac{Q_0}{K_0} L \end{aligned}$$

There's a contradiction here because C_1 has two values. Therefore, there is no equilibrium temperature distribution in the case that both ends of the rod are insulated. If there's a constant heat source in the rod and all parts of the rod are insulated, then we expect the temperature to rise indefinitely.

Part (b)

Bring ρ and c inside the time derivative.

$$\frac{\partial(\rho c u)}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q_0$$

The thermal energy density e is the product of mass density ρ, specific heat c, and temperature u.

$$\frac{\partial e}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q_0$$

To obtain the total thermal energy in the rod, integrate both sides over the rod's volume V.

$$\int_V \frac{\partial e}{\partial t}\, dV = \int_V \left(K_0 \frac{\partial^2 u}{\partial x^2} + Q_0\right) dV$$

Bring the time derivative in front of the integral on the left side. It becomes a total derivative, as the definite volume integral wipes out the x variable.

$$\frac{d}{dt}\int_V e\, dV = \int_V \left(K_0 \frac{\partial^2 u}{\partial x^2} + Q_0\right) dV$$

The integral on the left side represents the total thermal energy in the rod, and that's what we will solve for. The volume differential for the rod with constant cross-sectional area is $dV = A\, dx$.

$$\begin{aligned}
&= \int_0^L \left(K_0 \frac{\partial^2 u}{\partial x^2} + Q_0\right) A\, dx \\
&= A\left(K_0 \int_0^L \frac{\partial^2 u}{\partial x^2}\, dx + Q_0 \int_0^L dx\right) \\
&= A\left(K_0 \left.\frac{\partial u}{\partial x}\right|_0^L + Q_0 L\right) \\
&= A\left\{K_0 \left[\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t)\right] + Q_0 L\right\}
\end{aligned}$$

The fact that the rod is insulated at both ends means that $\partial u/\partial x(L,t) = \partial u/\partial x(0,t) = 0$.

$$\frac{d}{dt}\int_V e\, dV = Q_0 A L$$

Integrate both sides with respect to t.

$$\int_V e\, dV = Q_0 A L t + U_0$$

The aim now is to evaluate the constant of integration U_0, which represents the initial thermal energy in the rod. To do so, change e back in terms of u.

$$\int_V \rho c u\, dV = Q_0 A L t + U_0$$

Express the volume integral as one over the rod's length.

$$\int_0^L \rho c u A\, dx = Q_0 A L t + U_0$$

Bring the constants in front of the integral.

$$\rho c A \int_0^L u(x,t)\, dx = Q_0 A L t + U_0$$

Now set $t = 0$ in the equation.

$$\rho c A \int_0^L u(x,0)\, dx = U_0$$

Substitute the prescribed initial condition, $u(x, 0) = f(x)$.

$$\rho c A \int_0^L f(x)\, dx = U_0$$

Therefore, the total thermal energy in the rod is

$$\int_V e\, dV = Q_0 A L t + \rho c A \int_0^L f(x)\, dx.$$