Exercise 1.5.14

Isobars are lines of constant temperature. Show that isobars are perpendicular to any part of the boundary that is insulated.

[TYPO: Isobars are lines of constant pressure. Isotherms are lines of constant temperature.]

Solution

Isotherms are constant solutions (also known as level surfaces) to the heat equation.

$$u(x, y, z, t) = u_0$$

For fixed time, consider the differential of both sides.

$$du = 0$$

For a function with three spatial variables, the differential can be written like so.

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz = 0$$

Notice that the left side is a dot product of two vectors, ∇u and $\langle dx, dy, dz\rangle$. $\langle dx, dy, dz\rangle$ represents an arbitrary vector in the plane of constant temperature u_0.

$$\nabla u \cdot \langle dx, dy, dz\rangle = 0$$

Along an insulated boundary that has a unit normal vector $\boldsymbol{n}$, the temperature satisfies

$$\nabla u \cdot \boldsymbol{n} = 0.$$

Comparing the previous two equations, we conclude that $\boldsymbol{n}$ lies in the plane of constant temperature. That is, any line of constant temperature will run perpendicular to an insulated boundary.

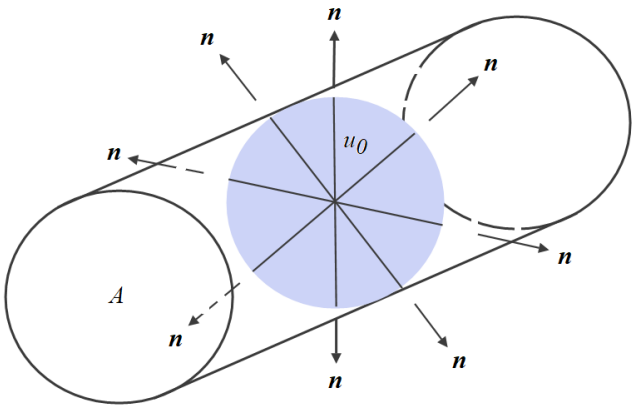

Figure 1: This is an illustration of an insulated one-dimensional rod. Any cross-section of the rod has a constant temperature, and all of the lines in this cross-section run perpendicular to the boundary (parallel to $\mathbf{n}$).