Exercise 1.5.14

Isobars are lines of constant temperature. Show that isobars are perpendicular to any part of the boundary that is insulated.

[TYPO: Isobars are lines of constant pressure. Isotherms are lines of constant temperature.]

Solution

Isotherms are constant solutions (also known as level surfaces) to the heat equation.

\[ u(x, y, z, t) = u_0 \]

For fixed time, consider the differential of both sides.

\[ du = 0 \]

For a function with three spatial variables, the differential can be written like so.

\[ \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \]

Notice that the left side is a dot product of two vectors, \( \nabla u \) and \( \langle dx, dy, dz \rangle \). \( \langle dx, dy, dz \rangle \) represents an arbitrary vector in the plane of constant temperature \( u_0 \).

\[ \nabla u \cdot \langle dx, dy, dz \rangle = 0 \]

Along an insulated boundary that has a unit normal vector \( n \), the temperature satisfies

\[ \nabla u \cdot n = 0. \]

Comparing the previous two equations, we conclude that \( n \) lies in the plane of constant temperature. That is, any line of constant temperature will run perpendicular to an insulated boundary.

Figure 1: This is an illustration of an insulated one-dimensional rod. Any cross-section of the rod has a constant temperature, and all of the lines in this cross-section run perpendicular to the boundary (parallel to \( n \)).