Exercise 1.5.25

Suppose a sphere of radius 2 satisfies \( \frac{\partial u}{\partial t} = \nabla^2 u + 5 \) with \( u(x, y, z, 0) = f(x, y, z) \) and on the surface of the sphere it is given that \( \nabla u \cdot \hat{n} = 6 \), where \( \hat{n} \) is a unit outward normal vector. Calculate the total thermal energy for this sphere as a function of time. (Hint: Use the divergence theorem.)

Solution

The governing equation for the sphere’s temperature \( u \) is

\[
\frac{\partial u}{\partial t} = \nabla^2 u + 5.
\]

Comparing this to the general form of the heat equation, we see that the mass density \( \rho \) and specific heat \( c \) are equal to 1 and that the heat source is \( Q = 5 \). The thermal energy density \( e \) is \( \rho c u = u \), so the left side can be written in terms of \( e \).

\[
\frac{\partial e}{\partial t} = \nabla^2 u + 5
\]

To obtain the total thermal energy in the sphere, integrate both sides over the sphere’s volume \( V \).

\[
\iiint_V \frac{\partial e}{\partial t} \, dV = \iiint_V (\nabla^2 u + 5) \, dV
\]

Bring the time derivative in front of the volume integral on the left and split the volume integral on the right into two.

\[
\frac{d}{dt} \iiint_V e \, dV = \iiint_V \nabla^2 u \, dV + 5 \iiint_V dV
\]

The triple integral on the left represents the total thermal energy in the sphere. The second integral on the right side is the sphere’s volume.

\[
= \iiint_V \nabla \cdot \nabla u \, dV + 5 \cdot \frac{4}{3} \pi (2)^3
\]

Apply the divergence theorem here to the remaining triple integral. The volume integral becomes an integral over the sphere’s surface.

\[
= \iint_S \nabla u \cdot \hat{n} \, dS + \frac{160\pi}{3}
\]

Use the fact that \( \nabla u \cdot \hat{n} = 6 \).

\[
= \iint_S 6 \, dS + \frac{160\pi}{3}
\]

The closed surface integral is just 6 times the surface area of the sphere.

\[
= 6 \cdot 4\pi (2)^2 + \frac{160\pi}{3}
\]

\[
= 96\pi + \frac{160\pi}{3}
\]

\[
\frac{d}{dt} \iiint_V e \, dV = \frac{448\pi}{3}
\]
To solve for the total thermal energy, integrate both sides with respect to $t$.

$$ \iiint_V e \, dV = \frac{448\pi}{3} t + C $$

To determine $C$, set $t = 0$ and use the prescribed initial condition $u(x, y, z, 0) = f(x, y, z)$.

$$ C = \iiint_V e(x, y, z, 0) \, dV = \iiint_V u(x, y, z, 0) \, dV = \iiint_V f(x, y, z) \, dV = \iiint_V f(x, y, z) \, dx \, dy \, dz $$

Therefore, the total thermal energy in the sphere is

$$ \iiint_V e \, dV = \frac{448\pi}{3} t + \iiint_V f(x, y, z) \, dx \, dy \, dz. $$