Exercise 1.5.9

Determine the equilibrium temperature distribution inside a circular annulus \((r_1 \leq r \leq r_2)\):

(a) if the outer radius is at temperature \(T_2\) and the inner at \(T_1\)

(b) if the outer radius is insulated and the inner radius is at temperature \(T_1\)

Solution

The governing equation for the temperature in the annulus, assuming radial symmetry, is

\[
\frac{\partial u}{\partial t} = k \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \quad r_1 \leq r \leq r_2.
\]

At equilibrium the temperature does not change in time, so \(\partial u/\partial t\) vanishes. \(u\) is only a function of \(r\) now.

\[
0 = \frac{k}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) \rightarrow \frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) = 0
\]

To solve the differential equation, multiply both sides by \(r\).

\[
\frac{d}{dr} \left( r \frac{du}{dr} \right) = 0
\]

Integrate both sides with respect to \(r\).

\[
r \frac{du}{dr} = C_1
\]

Divide both sides by \(r\).

\[
\frac{du}{dr} = \frac{C_1}{r}
\]

Part (a)

Integrate both sides with respect to \(r\) once more.

\[
u(r) = C_1 \ln r + C_2
\]

Apply the boundary conditions here to determine \(C_1\) and \(C_2\).

\[
u(r_1) = C_1 \ln r_1 + C_2 = T_1 \tag{1}
\]

\[
u(r_2) = C_1 \ln r_2 + C_2 = T_2 \tag{2}
\]

Subtract equation (2) from equation (1) to get an equation for \(C_1\).

\[
C_1 (\ln r_1 - \ln r_2) = T_1 - T_2
\]

\[
C_1 \ln \frac{r_1}{r_2} = T_1 - T_2 \quad \rightarrow \quad C_1 = \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}}
\]

Solve equation (1) for \(C_2\).

\[
C_2 = T_1 - C_1 \ln r_1
\]

\[
= T_1 - \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln r_1
\]
The equilibrium temperature distribution is then

\[ u(r) = C_1 \ln r + C_2 \]
\[ = \frac{T_1 - T_2}{\ln \frac{r}{r_2}} \ln r + T_1 - \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln r_1 \]
\[ = T_1 + \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} (\ln r - \ln r_1) \]
\[ = T_1 + \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln \frac{r}{r_1} \]
\[ = T_1 + \frac{T_2 - T_1}{\ln \frac{r_1}{r_2}} \ln \frac{r}{r_1} \]

Therefore, the equilibrium temperature distribution is

\[ u(r) = T_1 + (T_2 - T_1) \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}}. \]

**Part (b)**

If the outer radius is insulated, then \( du/dr \) must be zero there.

\[ \frac{du}{dr}(r_2) = \frac{C_3}{r_2} = 0 \quad \rightarrow \quad C_3 = 0 \]

The differential equation becomes

\[ \frac{du}{dr} = 0. \]

Integrate both sides with respect to \( r \) once more.

\[ u(r) = C_4 \]

Apply the second boundary condition to determine \( C_4 \).

\[ u(r_1) = C_4 = T_1 \]

Therefore, the equilibrium temperature distribution is

\[ u(r) = T_1. \]