Exercise 2.3.1

For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(a) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$

(b) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} - v_0\dfrac{\partial u}{\partial x}$

(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(d) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)$

(e) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^4 u}{\partial x^4}$

(f) $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$

Solution

Part (a)

The PDE in question here is

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Assume a product solution of the form $u(r,t) = R(r)T(t)$.

$$\begin{aligned}
\frac{\partial}{\partial t}[R(r)T(t)] &= \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}[R(r)T(t)]\right) \\
R(r)T'(t) &= \frac{k}{r}\frac{\partial}{\partial r}\left[rR'(r)T(t)\right] \\
R(r)T'(t) &= \frac{kT(t)}{r}\frac{d}{dr}\left[rR'(r)\right]
\end{aligned}$$

Now separate variables in the equation: divide both sides by $kR(r)T(t)$ so that all constants and functions of t are on the left side and all functions of r are on the right side.

$$\frac{T'(t)}{kT(t)} = \frac{1}{rR(r)}\frac{d}{dr}\left[rR'(r)\right]$$

The only way a function of t can be equal to a function of r is if both are equal to a constant λ.

$$\frac{T'(t)}{kT(t)} = \frac{1}{rR(r)}\frac{d}{dr}\left[rR'(r)\right] = \lambda$$

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

$$\left.\begin{aligned}
\frac{T'(t)}{kT(t)} &= \lambda \\
\frac{1}{rR(r)}\frac{d}{dr}\left[rR'(r)\right] &= \lambda
\end{aligned}\right\}$$

Part (b)

The PDE in question here is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}.$$

Assume a product solution of the form $u(x,t) = X(x)T(t)$.

$$\begin{aligned}\frac{\partial}{\partial t}[X(x)T(t)] &= k\frac{\partial^2}{\partial x^2}[X(x)T(t)] - v_0\frac{\partial}{\partial x}[X(x)T(t)] \\ X(x)T'(t) &= kX''(x)T(t) - v_0X'(x)T(t)\end{aligned}$$

Now separate variables in the equation: divide both sides by $X(x)T(t)$ so that all functions of t are on the left side and all constants and functions of x are on the right side.

$$\frac{T'(t)}{T(t)} = \frac{kX''(x) - v_0X'(x)}{X(x)}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{T'(t)}{T(t)} = \frac{kX''(x) - v_0X'(x)}{X(x)} = \lambda$$

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

$$\left.\begin{aligned}\frac{T'(t)}{T(t)} &= \lambda \\ \frac{kX''(x) - v_0X'(x)}{X(x)} &= \lambda\end{aligned}\right\}$$

Part (c)

The PDE in question here is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Assume a product solution of the form $u(x,t) = X(x)Y(y)$.

$$\begin{aligned}\frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] &= 0 \\ X''(x)Y(y) + X(x)Y''(y) &= 0\end{aligned}$$

Divide both sides by $X(x)Y(y)$.

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

Now separate variables in the equation: bring Y''/Y to the right side. Note that it doesn't matter which side the minus sign is placed on—the final answer for u will be the same either way.

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

$$\left.\begin{aligned} \frac{X''(x)}{X(x)} &= \lambda \\ -\frac{Y''(y)}{Y(y)} &= \lambda \end{aligned}\right\}$$

Part (d)

The PDE in question here is

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right).$$

Assume a product solution of the form $u(r,t) = R(r)T(t)$.

$$\begin{aligned} \frac{\partial}{\partial t}[R(r)T(t)] &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}[R(r)T(t)]\right) \\ R(r)T'(t) &= \frac{k}{r^2}\frac{\partial}{\partial r}\left[r^2 R'(r)T(t)\right] \\ R(r)T'(t) &= \frac{kT(t)}{r^2}\frac{d}{dr}\left[r^2 R'(r)\right] \end{aligned}$$

Now separate variables in the equation: divide both sides by $kR(r)T(t)$ so that all constants and functions of t are on the left side and all functions of r are on the right side.

$$\frac{T'(t)}{kT(t)} = \frac{1}{r^2R(r)}\frac{d}{dr}\left[r^2R'(r)\right]$$

The only way a function of t can be equal to a function of r is if both are equal to a constant λ.

$$\frac{T'(t)}{kT(t)} = \frac{1}{r^2R(r)}\frac{d}{dr}\left[r^2R'(r)\right] = \lambda$$

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

$$\left.\begin{aligned} \frac{T'(t)}{kT(t)} &= \lambda \\ \frac{1}{r^2R(r)}\frac{d}{dr}\left[r^2R'(r)\right] &= \lambda \end{aligned}\right\}$$

Part (e)

The PDE in question here is

$$\frac{\partial u}{\partial t} = k\frac{\partial^4 u}{\partial x^4}.$$

Assume a product solution of the form $u(x,t) = X(x)T(t)$.

$$\frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^4}{\partial x^4}[X(x)T(t)]$$
$$X(x)T'(t) = kX''''(x)T(t)$$

Now separate variables in the equation: divide both sides by $kX(x)T(t)$ so that all constants and functions of t are on the left side and all functions of x are on the right side.

$$\frac{T'(t)}{kT(t)} = \frac{X''''(x)}{X(x)}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{T'(t)}{kT(t)} = \frac{X''''(x)}{X(x)} = \lambda$$

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

$$\left.\begin{aligned}\frac{T'(t)}{kT(t)} &= \lambda \\ \frac{X''''(x)}{X(x)} &= \lambda\end{aligned}\right\}$$

Part (f)

The PDE in question here is

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

Assume a product solution of the form $u(x,t) = X(x)T(t)$.

$$\frac{\partial^2}{\partial t^2}[X(x)T(t)] = c^2\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$
$$X(x)T''(t) = c^2X''(x)T(t)$$

Now separate variables in the equation: divide both sides by $c^2X(x)T(t)$ so that all constants and functions of t are on the left side and all functions of x are on the right side.

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

$$\left.\begin{aligned} \frac{T''(t)}{c^2T(t)} &= \lambda \\ \frac{X''(x)}{X(x)} &= \lambda \end{aligned}\right\}$$