Exercise 2.2.1

Show that any linear combination of linear operators is a linear operator.

Solution

Suppose $L_1$ and $L_2$ are linear operators. Then, by the definition of linearity,

\[
L_1(c_1 u_1 + c_2 u_2) = c_1 L_1(u_1) + c_2 L_1(u_2) \\
L_2(c_1 u_1 + c_2 u_2) = c_1 L_2(u_1) + c_2 L_2(u_2),
\]

where $c_1$ and $c_2$ are arbitrary constants and $u_1$ and $u_2$ are solutions to a linear homogeneous equation. The aim is to show that a linear combination of $L_1$ and $L_2$, $c_3 L_1 + c_4 L_2$, is also linear.

\[
(c_3 L_1 + c_4 L_2)(c_1 u_1 + c_2 u_2) = c_1(c_3 L_1 + c_4 L_2)(u_1) + c_2(c_3 L_1 + c_4 L_2)(u_2).
\]

We have

\[
(c_3 L_1 + c_4 L_2)(c_1 u_1 + c_2 u_2) = c_3 L_1(c_1 u_1 + c_2 u_2) + c_4 L_2(c_1 u_1 + c_2 u_2) \\
= c_3 [c_1 L_1(u_1) + c_2 L_1(u_2)] + c_4 [c_1 L_2(u_1) + c_2 L_2(u_2)] \\
= c_3 c_1 L_1(u_1) + c_3 c_2 L_1(u_2) + c_4 c_1 L_2(u_1) + c_4 c_2 L_2(u_2) \\
= c_1 c_3 L_1(u_1) + c_1 c_4 L_1(u_2) + c_2 c_3 L_1(u_2) + c_2 c_4 L_2(u_2) \\
= c_1 [c_3 L_1(u_1) + c_4 L_2(u_1)] + c_2 [c_3 L_1(u_2) + c_4 L_2(u_2)] \\
= c_1 (c_3 L_1 + c_4 L_2)(u_1) + c_2 (c_3 L_1 + c_4 L_2)(u_2).
\]

Therefore, any linear combination of linear operators is a linear operator.