Exercise 2.2.3

Show that \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(u, x, t) \) is linear if \( Q = \alpha(x, t)u + \beta(x, t) \) and, in addition, homogeneous if \( \beta(x, t) = 0 \).

Solution

Suppose that \( Q = \alpha(x, t)u + \beta(x, t) \). The differential equation becomes

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + \beta(x, t).
\]

Isolate \( \beta(x, t) \) on the right side.

\[
\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} - \alpha(x, t)u = \beta(x, t)
\]

Write the left side as an operator \( L \) acting on \( u \).

\[
\left[ \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} - \alpha(x, t) \right] u = \beta(x, t)
\]

\( L(u) = \beta(x, t) \)

Notice that the PDE is homogeneous if \( \beta(x, t) = 0 \). The aim now is to show that \( L \) is linear,

\[
L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2),
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants and \( u_1 \) and \( u_2 \) are solutions to the PDE. We have

\[
L(c_1u_1 + c_2u_2) = \left[ \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} - \alpha(x, t) \right] (c_1u_1 + c_2u_2)
\]

\[
= \frac{\partial}{\partial t} (c_1u_1 + c_2u_2) - k \frac{\partial^2}{\partial x^2} (c_1u_1 + c_2u_2) - \alpha(x, t) (c_1u_1 + c_2u_2)
\]

\[
= c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t} - c_1 k \frac{\partial^2 u_1}{\partial x^2} - c_2 k \frac{\partial^2 u_2}{\partial x^2} - c_1 \alpha(x, t) u_1 - c_2 \alpha(x, t) u_2
\]

\[
= c_1 \left[ \frac{\partial u_1}{\partial t} - c_1 k \frac{\partial^2 u_1}{\partial x^2} - c_1 \alpha(x, t) u_1 \right] + c_2 \left[ \frac{\partial u_2}{\partial t} - c_2 k \frac{\partial^2 u_2}{\partial x^2} - c_2 \alpha(x, t) u_2 \right]
\]

\[
= c_1 \left[ \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} - \alpha(x, t) \right] u_1 + c_2 \left[ \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} - \alpha(x, t) \right] u_2
\]

\[
= c_1 L(u_1) + c_2 L(u_2).
\]

Therefore, \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(u, x, t) \) is linear if \( Q = \alpha(x, t)u + \beta(x, t) \) and, in addition, homogeneous if \( \beta(x, t) = 0 \).