Exercise 2.2.4

In this exercise we derive superposition principles for nonhomogeneous problems.

(a) Consider $L(u) = f$. If $u_p$ is a particular solution, $L(u_p) = f$, and if $u_1$ and $u_2$ are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1 u_1 + c_2 u_2$ is another particular solution.

(b) If $L(u) = f_1 + f_2$, where $u_{p_i}$ is a particular solution corresponding to $f_i$, what is a particular solution for $f_1 + f_2$?

Solution

Part (a)

Here we have to show that

$$L(u_p + c_1 u_1 + c_2 u_2) = f.$$  

Use the fact that $L$ is a linear operator and simplify.

$$L(u_p + c_1 u_1 + c_2 u_2) = L(u_p) + c_1 L(u_1) + c_2 L(u_2)$$
$$= f + c_1 (0) + c_2 (0)$$
$$= f$$

Therefore, $u = u_p + c_1 u_1 + c_2 u_2$ is another particular solution.

Part (b)

$u_{p_i}$ is a particular solution corresponding to $f_i$, so we have the following equations to work with.

$$L(u_{p_1}) = f_1$$
$$L(u_{p_2}) = f_2$$

Add these two equations to get

$$L(u_{p_1}) + L(u_{p_2}) = f_1 + f_2.$$  

Use the fact that $L$ is linear.

$$L(u_{p_1} + u_{p_2}) = f_1 + f_2$$

Therefore, $u = u_{p_1} + u_{p_2}$ is a particular solution for $f_1 + f_2$.  

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