Exercise 2.3.1

For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(a) \[ \frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \]

(b) \[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x} \]

(c) \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

(d) \[ \frac{\partial u}{\partial t} = k \frac{\partial}{r^2} \left( r^2 \frac{\partial u}{\partial r} \right) \]

(e) \[ \frac{\partial u}{\partial t} = k \frac{\partial^4 u}{\partial x^4} \]

(f) \[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

Solution

Part (a)

The PDE in question here is

\[ \frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right). \]

Assume a product solution of the form \( u(r,t) = R(r)T(t) \).

\[ \frac{\partial}{\partial t} [R(r)T(t)] = \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial [R(r)T(t)]}{\partial r} \right) \]

\[ R(r)T'(t) = \frac{k}{r} \frac{\partial}{\partial r} \left[ rR'(r)T(t) \right] \]

\[ R(r)T'(t) = \frac{kT(t)}{r} \frac{d}{dr} \left[ rR'(r) \right] \]

Now separate variables in the equation: divide both sides by \( kR(r)T(t) \) so that all constants and functions of \( t \) are on the left side and all functions of \( r \) are on the right side.

\[ \frac{T'(t)}{kT(t)} = \frac{1}{rR(r)} \frac{d}{dr} \left[ rR'(r) \right] \]

The only way a function of \( t \) can be equal to a function of \( r \) is if both are equal to a constant \( \lambda \).

\[ \frac{T'(t)}{kT(t)} = \frac{1}{rR(r)} \frac{d}{dr} \left[ rR'(r) \right] = \lambda \]

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

\[ \begin{cases} \frac{T'(t)}{kT(t)} = \lambda \\ \frac{1}{rR(r)} \frac{d}{dr} \left[ rR'(r) \right] = \lambda \end{cases} \]
Part (b)

The PDE in question here is
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x}. \]

Assume a product solution of the form \( u(x, t) = X(x)T(t) \).

\[ \frac{\partial}{\partial t} [X(x)T(t)] = k \frac{\partial^2}{\partial x^2} [X(x)T(t)] - v_0 \frac{\partial}{\partial x} [X(x)T(t)] \]
\[ X(x)T'(t) = kX''(x)T(t) - v_0 X'(x)T(t) \]

Now separate variables in the equation: divide both sides by \( X(x)T(t) \) so that all functions of \( t \) are on the left side and all constants and functions of \( x \) are on the right side.
\[ \frac{T'(t)}{T(t)} = \frac{kX''(x) - v_0 X'(x)}{X(x)} \]

The only way a function of \( t \) can be equal to a function of \( x \) is if both are equal to a constant \( \lambda \).
\[ \frac{T'(t)}{T(t)} = \frac{kX''(x) - v_0 X'(x)}{X(x)} = \lambda \]

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.
\[ \begin{align*}
    \frac{T'(t)}{T(t)} &= \lambda \\
    \frac{kX''(x) - v_0 X'(x)}{X(x)} &= \lambda
\end{align*} \]

Part (c)

The PDE in question here is
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \]

Assume a product solution of the form \( u(x, t) = X(x)Y(y) \).

\[ \frac{\partial^2}{\partial x^2} [X(x)Y(y)] + \frac{\partial^2}{\partial y^2} [X(x)Y(y)] = 0 \]
\[ X''(x)Y(y) + X(x)Y''(y) = 0 \]

Divide both sides by \( X(x)Y(y) \).
\[ \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \]

Now separate variables in the equation: bring \( Y''/Y \) to the right side. Note that it doesn’t matter which side the minus sign is placed on—the final answer for \( u \) will be the same either way.
\[ \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \]
The only way a function of \( x \) can be equal to a function of \( y \) is if both are equal to a constant \( \lambda \).

\[
\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda
\]

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

\[
\begin{aligned}
\frac{X''(x)}{X(x)} &= \lambda \\
\frac{Y''(y)}{Y(y)} &= \lambda
\end{aligned}
\]

**Part (d)**

The PDE in question here is

\[
\frac{\partial u}{\partial t} = k \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right).
\]

Assume a product solution of the form \( u(r,t) = R(r)T(t) \).

\[
\frac{\partial}{\partial t} [R(r)T(t)] = k \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} [R(r)T(t)] \right)
\]

\[
R(r)T'(t) = k \frac{\partial}{\partial r} [r^2 R(r)T(t)]
\]

\[
R(r)T'(t) = \frac{kT'(t)}{r^2} \frac{d}{dr} [r^2 R'(r)]
\]

Now separate variables in the equation: divide both sides by \( kR(r)\) \( T(t) \) so that all constants and functions of \( t \) are on the left side and all functions of \( r \) are on the right side.

\[
\frac{T'(t)}{kT(t)} = \frac{1}{r^2 R(r)} \frac{d}{dr} [r^2 R'(r)]
\]

The only way a function of \( t \) can be equal to a function of \( r \) is if both are equal to a constant \( \lambda \).

\[
\frac{T'(t)}{kT(t)} = \lambda \]

\[
\frac{1}{r^2 R(r)} \frac{d}{dr} [r^2 R'(r)] = \lambda
\]

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

\[
\begin{aligned}
\frac{T'(t)}{kT(t)} &= \lambda \\
\frac{1}{r^2 R(r)} \frac{d}{dr} [r^2 R'(r)] &= \lambda
\end{aligned}
\]
Part (e)

The PDE in question here is

$$\frac{\partial u}{\partial t} = k \frac{\partial^4 u}{\partial x^4}$$

Assume a product solution of the form \( u(x,t) = X(x)T(t) \).

$$\frac{\partial}{\partial t} [X(x)T(t)] = k \frac{\partial^4}{\partial x^4} [X(x)T(t)]$$

$$X(x)T'(t) = kX^{''''}(x)T(t)$$

Now separate variables in the equation: divide both sides by \( kX(x)T(t) \) so that all constants and functions of \( t \) are on the left side and all functions of \( x \) are on the right side.

$$\frac{T'(t)}{kT(t)} = \frac{X^{'''}(x)}{X(x)}$$

The only way a function of \( t \) can be equal to a function of \( x \) is if both are equal to a constant \( \lambda \).

$$\frac{T'(t)}{kT(t)} = \frac{X^{'''}(x)}{X(x)} = \lambda$$

As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

$$\begin{cases} 
\frac{T'(t)}{kT(t)} = \lambda \\
\frac{X^{'''}(x)}{X(x)} = \lambda 
\end{cases}$$

Part (f)

The PDE in question here is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Assume a product solution of the form \( u(x,t) = X(x)T(t) \).

$$\frac{\partial^2}{\partial t^2} [X(x)T(t)] = c^2 \frac{\partial^2}{\partial x^2} [X(x)T(t)]$$

$$X(x)T''(t) = c^2 X''(x)T(t)$$

Now separate variables in the equation: divide both sides by \( c^2X(x)T(t) \) so that all constants and functions of \( t \) are on the left side and all functions of \( x \) are on the right side.

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}$$

The only way a function of \( t \) can be equal to a function of \( x \) is if both are equal to a constant \( \lambda \).

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = \lambda$$
As a result of the method of separation of variables, the PDE has reduced to a system of ODEs, one in each independent variable.

\[
\begin{align*}
\frac{T''(t)}{c^2 T(t)} &= \lambda \\
\frac{X''(x)}{X(x)} &= \lambda
\end{align*}
\]