Exercise 2.3.11

Solve the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to the following conditions:

$$u(0,t) = 0 \quad u(L,t) = 0 \quad u(x,0) = f(x).$$

What happens as $t \to \infty$? [*Hints*:

1. It is known that if $u(x,t) = \phi(x)G(t)$, then $\dfrac{1}{kG}\dfrac{dG}{dt} = \dfrac{1}{\phi}\dfrac{d^2\phi}{dx^2}$.
2. Use formula sheet.]

Solution

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \to \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned} u(0,t) &= 0 & &\to & X(0)T(t) &= 0 & &\to & X(0) &= 0 \\ u(L,t) &= 0 & &\to & X(L)T(t) &= 0 & &\to & X(L) &= 0 \end{aligned}$$

Now separate variables in the PDE.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. (Note that the final answer for u will be the same regardless which side k is on. Normally constants are grouped with t.)

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} &= \lambda \\ \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(L) &= C_1 \cosh \alpha L + C_2 \sinh \alpha L = 0 \end{aligned}$$

The second equation reduces to $C_2 \sinh \alpha L = 0$. Because hyperbolic sine is not oscillatory, C_2 must be zero for the equation to be satisfied. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions now to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(L) &= C_3 L + C_4 = 0 \end{aligned}$$

The second equation reduces to $C_3 = 0$. This results in the trivial solution $X(x) = 0$, which means zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(L) &= C_5 \cos \beta L + C_6 \sin \beta L = 0 \end{aligned}$$

The second equation reduces to $C_6 \sin \beta L = 0$. To avoid the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

There are negative eigenvalues $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos \beta x + C_6 \sin \beta x \\ &= C_6 \sin \beta x \quad \to \quad X_n(x) = \sin \frac{n\pi x}{L}. \end{aligned}$$

n only takes on the values it does because negative integers result in redundant values for λ. With this formula for λ, the ODE for T becomes

$$\frac{1}{kT}\frac{dT}{dt} = -\frac{n^2\pi^2}{L^2}.$$

Multiply both sides by kT.

$$\frac{dT}{dt} = -\frac{kn^2\pi^2}{L^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \quad \to \quad T_n(t) = \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ for each of the eigenvalues.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \sin \frac{n\pi x}{L}}$$

Apply the initial condition $u(x, 0) = f(x)$ now to determine B_n.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x)$$

Multiply both sides by $\sin(m\pi x/L)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} = f(x) \sin \frac{m\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$$

Bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$B_n \int_0^L \sin^2 \frac{n\pi x}{L}\, dx = \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Evaluate the integral on the left.

$$B_n\left(\frac{L}{2}\right) = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

So then

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.}$$

Because of the decaying exponential function, u falls to zero as $t \to \infty$.

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\sum_{n=1}^{\infty} B_n \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\sin\frac{n\pi x}{L} \\ &= 0\end{aligned}$$