Exercise 2.3.3

Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\frac{9\pi x}{L}$ **(b)** $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$

(c) $u(x,0) = 2\cos\frac{3\pi x}{L}$ **(d)** $u(x,0) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases}$

(e) $u(x,0) = f(x)$

[Your answer in part (c) may involve certain integrals that do not need to be evaluated.]

Solution

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \to \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned} u(0,t) = 0 \quad &\to \quad X(0)T(t) = 0 \quad &\to \quad X(0) = 0 \\ u(L,t) = 0 \quad &\to \quad X(L)T(t) = 0 \quad &\to \quad X(L) = 0 \end{aligned}$$

Now separate variables in the PDE.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. Note that the final answer for u will be the same regardless which side k is on. Constants are normally grouped with t.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} &= \lambda \\ \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(L) &= C_1 \cosh \alpha L + C_2 \sinh \alpha L = 0 \end{aligned}$$

The second equation reduces to $C_2 \sinh \alpha L = 0$. Because hyperbolic sine is not oscillatory, C_2 must be zero for the equation to be satisfied. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions now to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(L) &= C_3 L + C_4 = 0 \end{aligned}$$

The second equation reduces to $C_3 = 0$. This results in the trivial solution $X(x) = 0$, which means zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(L) &= C_5 \cos \beta L + C_6 \sin \beta L = 0 \end{aligned}$$

The second equation reduces to $C_6 \sin \beta L = 0$. To avoid the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

There are negative eigenvalues $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos\beta x + C_6 \sin\beta x \\ &= C_6 \sin\beta x \quad \rightarrow \quad X_n(x) = \sin\frac{n\pi x}{L}. \end{aligned}$$

n only takes on the values it does because negative integers result in redundant values for λ. With this formula for λ, the ODE for T becomes

$$\frac{1}{kT}\frac{dT}{dt} = -\frac{n^2\pi^2}{L^2}.$$

Multiply both sides by kT.

$$\frac{dT}{dt} = -\frac{kn^2\pi^2}{L^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \quad \rightarrow \quad T_n(t) = \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\sin\frac{n\pi x}{L}$$

Apply the initial condition now to determine B_n.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

Part (a)

Here the initial condition is $u(x,0) = 6\sin\frac{9\pi x}{L}$.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = 6\sin\frac{9\pi x}{L}$$

By inspection we see that

$$B_n = \begin{cases} 0 & \text{if } n \neq 9 \\ 6 & \text{if } n = 9 \end{cases}.$$

The general solution for u reduces to

$$u(x,t) = B_9 \exp\left(-\frac{k(9)^2\pi^2}{L^2}t\right)\sin\frac{9\pi x}{L}.$$

Therefore,

$$u(x,t) = 6\exp\left(-\frac{81\pi^2 k}{L^2}t\right)\sin\frac{9\pi x}{L}.$$

Part (b)

Here the initial condition is $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$$

By inspection we see that

$$B_n = \begin{cases} 0 & \text{if } n \neq 1,\ n \neq 3 \\ 3 & \text{if } n = 1 \\ -1 & \text{if } n = 3 \end{cases}.$$

The general solution for u reduces to

$$u(x,t) = B_1 \exp\left(-\frac{k(1)^2\pi^2}{L^2}t\right)\sin\frac{\pi x}{L} + B_3 \exp\left(-\frac{k(3)^2\pi^2}{L^2}t\right)\sin\frac{3\pi x}{L}.$$

Therefore,

$$u(x,t) = 3\exp\left(-\frac{\pi^2 k}{L^2}t\right)\sin\frac{\pi x}{L} - \exp\left(-\frac{9\pi^2 k}{L^2}t\right)\sin\frac{3\pi x}{L}.$$

Part (c)

Here the initial condition is $u(x,0) = 2\cos\frac{3\pi x}{L}$.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = 2\cos\frac{3\pi x}{L}$$

Multiply both sides by $\sin(m\pi x/L)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = 2\cos\frac{3\pi x}{L}\sin\frac{m\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{m\pi x}{L}\,dx$$

Bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = 2\int_0^L \cos\frac{3\pi x}{L}\sin\frac{m\pi x}{L}\,dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$B_n \int_0^L \sin^2\frac{n\pi x}{L}\,dx = 2\int_0^L \cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

Note that if $n = 3$, then the integral on the right is zero because sine and cosine are orthogonal: $B_3 = 0$. Use the power-reducing formula for sine on the left and the product-to-sum formula for cosine-sine on the right.

$$B_n \int_0^L \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{L}\right) dx = 2\int_0^L \frac{1}{2}\left[\sin\left(\frac{3\pi x}{L} + \frac{n\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L} - \frac{n\pi x}{L}\right)\right]$$

Evaluate the integrals.

$$\begin{aligned} B_n\left(\frac{L}{2}\right) &= -\frac{L}{(3+n)\pi}\cos\frac{(3+n)\pi x}{L}\bigg|_0^L + \frac{L}{(3-n)\pi}\cos\frac{(3-n)\pi x}{L}\bigg|_0^L \\ &= -\frac{L}{(3+n)\pi}[\cos(3\pi + n\pi) - 1] + \frac{L}{(3-n)\pi}[\cos(3\pi - n\pi) - 1] \\ &= \frac{2nL}{(n^2-9)\pi}[1 + (-1)^n] \end{aligned}$$

$$B_n = \frac{4n}{(n^2-9)\pi}[1 + (-1)^n]$$

Notice that B_n simplifies if n is even or odd.

$$B_n = \begin{cases} 0 & \text{if } n = 2p - 1 \\ \dfrac{4(2p)}{[(2p)^2 - 9]\pi}(2) & \text{if } n = 2p,\ p = 1, 2, \ldots \end{cases}$$

The general solution for u reduces to

$$\begin{aligned} u(x,t) &= \sum_{2p=2}^{\infty} B_{2p} \exp\left(-\frac{k(2p)^2\pi^2}{L^2}t\right)\sin\frac{(2p)\pi x}{L} \\ &= \sum_{p=1}^{\infty} \frac{16p}{(4p^2-9)\pi}\exp\left(-\frac{4kp^2\pi^2}{L^2}t\right)\sin\frac{2p\pi x}{L}. \end{aligned}$$

Therefore,

$$u(x,t) = \frac{16}{\pi}\sum_{p=1}^{\infty}\frac{p}{4p^2-9}\exp\left(-\frac{4kp^2\pi^2}{L^2}t\right)\sin\frac{2p\pi x}{L}.$$

Part (d)

Here the initial condition is $u(x,0) = 1$ if $0 < x \le L/2$ and $u(x,0) = 2$ if $L/2 < x < L$.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = g(x) = \begin{cases} 1 & \text{if } 0 < x \le L/2 \\ 2 & \text{if } L/2 < x < L \end{cases}$$

Multiply both sides by $\sin(m\pi x/L)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = g(x)\sin\frac{m\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L g(x)\sin\frac{m\pi x}{L}\,dx$$

Bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L g(x)\sin\frac{m\pi x}{L}\,dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$B_n \int_0^L \sin^2\frac{n\pi x}{L}\,dx = \int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

Use the power-reducing formula for sine on the left and the product-to-sum formula for cosine-sine on the right.

$$B_n \int_0^L \frac{1}{2}\left(1-\cos\frac{2n\pi x}{L}\right)dx = \int_0^{L/2} \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^L 2\sin\frac{n\pi x}{L}\,dx$$

Evaluate the integrals.

$$B_n\left(\frac{L}{2}\right) = \frac{2L}{n\pi}\sin^2\frac{n\pi}{4} + \frac{2L}{n\pi}\left[\cos\frac{n\pi}{2} - (-1)^n\right]$$

So then

$$B_n = \frac{4}{n\pi}\left[\sin^2\frac{n\pi}{4} + \cos\frac{n\pi}{2} - (-1)^n\right].$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\left[\sin^2\frac{n\pi}{4} + \cos\frac{n\pi}{2} - (-1)^n\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\sin\frac{n\pi x}{L}.$$

Part (e)

Here the initial condition is $u(x,0) = f(x)$.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$$

Multiply both sides by $\sin(m\pi x/L)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = f(x)\sin\frac{m\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$B_n \int_0^L \sin^2 \frac{n\pi x}{L} \, dx = \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Evaluate the integral on the left.

$$B_n \left(\frac{L}{2}\right) = \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

So then

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

Therefore, changing the dummy integration variable to r,

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{L} \int_0^L f(r) \sin \frac{n\pi r}{L} \, dr\right] \exp\left(-\frac{kn^2\pi^2}{L^2} t\right) \sin \frac{n\pi x}{L}.$$