Exercise 2.3.4

Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0,t) = 0$, $u(L,t) = 0$, and $u(x,0) = f(x)$.

(a) What is the total heat energy in the rod as a function of time?

(b) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

(c) What relationship should exist between parts (a) and (b)?

Solution

The solution to this initial boundary value problem was found in Exercise 2.3.3(e).

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\sin\frac{n\pi x}{L}$$

Note that k is the thermal diffusivity and that

$$k = \frac{K_0}{\rho c},$$

where K_0 is the thermal conductivity, ρ is the mass density, and c is the specific heat.

Part (a)

The total heat energy in the rod is obtained by integrating the thermal energy density $e(x,t)$ over the rod's volume V. (A is the rod's cross-sectional area.)

$$\begin{aligned}
q(t) &= \int_V e(x,t)\,dV \\
&= \int_0^L e(x,t)(A\,dx) \\
&= \int_0^L \rho c u(x,t)(A\,dx) \\
&= \rho c A\int_0^L u(x,t)\,dx \\
&= \rho c A\int_0^L \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\sin\frac{n\pi x}{L}\,dx \\
&= \rho c A\sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\int_0^L \sin\frac{n\pi x}{L}\,dx \\
&= \rho c A\sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\frac{L[1-(-1)^n]}{n\pi} \\
&= \frac{2\rho c A}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)
\end{aligned}$$

Notice that if n is even, then the summand is zero. This formula can thus be simplified (that is, made to converge faster) by summing over the odd integers only. Substitute $n = 2m-1$ in the sum, where m is another integer.

$$q(t) = \frac{2\rho c A}{\pi} \sum_{2m-1=1}^{\infty} \left[\frac{2}{2m-1} \int_0^L f(r) \sin\frac{(2m-1)\pi r}{L}\, dr\right] \exp\left(-\frac{k(2m-1)^2\pi^2}{L^2}t\right)$$

Therefore,

$$q(t) = \frac{4\rho c A}{\pi} \sum_{m=1}^{\infty} \left[\frac{1}{2m-1} \int_0^L f(r) \sin\frac{(2m-1)\pi r}{L}\, dr\right] \exp\left(-\frac{k(2m-1)^2\pi^2}{L^2}t\right).$$

Part (b)

According to Fourier's law of conduction, the heat flux is

$$\phi = -K_0 \frac{\partial u}{\partial x}.$$

Assuming that the temperature $u(x,t)$ is continuous, the infinite series can in fact be differentiated term-by-term because $u(0,t) = 0$ and $u(L,t) = 0$. The heat fluxes at $x = 0$ and $x = L$ are then

$$\begin{aligned}
\phi|_{x=0} &= -K_0 \frac{\partial u}{\partial x}\bigg|_{x=0} \\
&= -K_0 \frac{\partial}{\partial x} \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right] \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \sin\frac{n\pi x}{L}\bigg|_{x=0} \\
&= -K_0 \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right] \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \frac{\partial}{\partial x}\sin\frac{n\pi x}{L}\bigg|_{x=0} \\
&= -K_0 \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right] \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \left(\frac{n\pi}{L}\right) \\
&= -\frac{2\pi K_0}{L^2} \sum_{n=1}^{\infty} \left[n\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right] \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \\
\phi|_{x=L} &= -K_0 \frac{\partial u}{\partial x}\bigg|_{x=L} \\
&= -K_0 \frac{\partial}{\partial x} \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right] \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \sin\frac{n\pi x}{L}\bigg|_{x=L} \\
&= -K_0 \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right] \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \frac{\partial}{\partial x}\sin\frac{n\pi x}{L}\bigg|_{x=L} \\
&= -K_0 \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right] \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \left[(-1)^n\frac{n\pi}{L}\right] \\
&= \frac{2\pi K_0}{L^2} \sum_{n=1}^{\infty} \left[n(-1)^{n+1}\int_0^L f(r)\sin\frac{n\pi r}{L}\,dr\right] \exp\left(-\frac{kn^2\pi^2}{L^2}t\right).
\end{aligned}$$

Part (c)

The relationship between the results in part (a) and part (b) is obtained by integrating both sides of the PDE with respect to x from 0 to L.

$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L k\frac{\partial^2 u}{\partial x^2}\,dx$$

Bring the time derivative outside the integral on the left and evaluate the integral on the right.

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)\,dx &= k\frac{\partial u}{\partial x}\bigg|_{x=0}^{x=L} \\ &= k\left[\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t)\right] \\ &= \frac{K_0}{\rho c}\left[\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t)\right]\end{aligned}$$

Multiply both sides by $\rho c A$ and distribute K_0.

$$\rho c A\frac{d}{dt}\int_0^L u(x,t)\,dx = A\left[K_0\frac{\partial u}{\partial x}(L,t) - K_0\frac{\partial u}{\partial x}(0,t)\right]$$

$$\frac{d}{dt}\left[\rho c A\int_0^L u(x,t)\,dx\right] = A\left[-K_0\frac{\partial u}{\partial x}(0,t)\right] - A\left[-K_0\frac{\partial u}{\partial x}(L,t)\right]$$

Therefore,

$$\frac{dq}{dt} = A\phi|_{x=0} - A\phi|_{x=L}\,.$$