Exercise 2.3.7

Consider the following boundary value problem (if necessary, see Section 2.4.1):

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x). \]

(a) Give a one-sentence physical interpretation of this problem.

(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. \[ \text{[Hint: The answer is} \]

\[ u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n kt} \cos \frac{n\pi x}{L}. \]

What is \( \lambda_n \)?

(c) Show that the initial condition, \( u(x,0) = f(x) \), is satisfied if

\[ f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}. \]

(d) Using Exercise 2.3.6, solve for \( A_0 \) and \( A_n \) (\( n \geq 1 \)).

(e) What happens to the temperature distribution as \( t \rightarrow \infty \)? Show that it approaches the steady-state temperature distribution (see Section 1.4).

Solution

Part (a)

The PDE is the governing equation for the temperature in a one-dimensional rod that is homogeneous and has constant cross-sectional area. The boundary conditions indicate that the rod is insulated at the \( x = 0 \) and \( x = L \) ends. Initially the temperature distribution in the rod is \( u(x,0) = f(x) \).

Part (b)

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form \( u(x,t) = X(x)T(t) \) and substitute it into the PDE

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad \frac{\partial}{\partial t}[X(x)T(t)] = k \frac{\partial^2}{\partial x^2}[X(x)T(t)] \]

and the boundary conditions.

\[ \frac{\partial u}{\partial x}(0,t) = 0 \quad \rightarrow \quad X'(0)T(t) = 0 \quad \rightarrow \quad X'(0) = 0 \]

\[ \frac{\partial u}{\partial x}(L,t) = 0 \quad \rightarrow \quad X'(L)T(t) = 0 \quad \rightarrow \quad X'(L) = 0 \]

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Now separate variables in the PDE.

$$X \frac{dT}{dt} = kT \frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. Note that the final answer for $u$ will be the same regardless which side $k$ is on. Constants are normally grouped with $t$.

$$\frac{1}{kT} \frac{dT}{dt} = \frac{1}{X} \frac{d^2X}{dx^2} = \lambda$$

The only way a function of $t$ can be equal to a function of $x$ is if both are equal to a constant $\lambda$.

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in $x$ and one in $t$.

Values of $\lambda$ that result in nontrivial solutions for $X$ and $T$ are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that $\lambda$ is positive: $\lambda = \alpha^2$.

The ODE for $X$ becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative with respect to $x$.

$$X'(x) = \alpha (C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions now to determine $C_1$ and $C_2$.

$$X'(0) = \alpha (C_2) = 0$$
$$X'(L) = \alpha (C_1 \sinh \alpha L + C_2 \cosh \alpha L) = 0$$

The first equation implies that $C_2 = 0$, so the second equation reduces to $C_1 \alpha \sinh \alpha L = 0$.

Because hyperbolic sine is not oscillatory, $C_1$ must be zero for the equation to be satisfied. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that $\lambda$ is zero: $\lambda = 0$. The ODE for $X$ becomes

$$\frac{d^2X}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to $x$ twice.

$$\frac{dX}{dx} = C_3$$

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Apply the boundary conditions now.

\[ X'(0) = C_3 = 0 \]
\[ X'(L) = C_3 = 0 \]

Consequently,

\[ \frac{dX}{dx} = 0. \]

Integrate both sides with respect to \( x \) once more.

\[ X(x) = C_4 \]

Zero is an eigenvalue because \( X(x) \) is not zero. The eigenfunction associated with it is \( X_0(x) = 1 \).

Solve the ODE for \( T \) now with \( \lambda = 0 \).

\[ \frac{dT}{dt} = 0 \quad \rightarrow \quad T_0(t) = \text{constant} \]

Suppose thirdly that \( \lambda \) is negative: \( \lambda = -\beta^2 \). The ODE for \( X \) becomes

\[ \frac{d^2X}{dx^2} = -\beta^2 X. \]

The general solution is written in terms of sine and cosine.

\[ X(x) = C_5 \cos \beta x + C_6 \sin \beta x \]

Take a derivative of it with respect to \( x \).

\[ X'(x) = \beta(-C_5 \sin \beta x + C_6 \cos \beta x) \]

Apply the boundary conditions now to determine \( C_5 \) and \( C_6 \).

\[ X'(0) = \beta(C_6) = 0 \]
\[ X'(L) = \beta(-C_5 \sin \beta L + C_6 \cos \beta L) = 0 \]

The first equation implies that \( C_6 = 0 \), so the second equation reduces to \( -C_5 \beta \sin \beta L = 0 \). To avoid the trivial solution, we insist that \( C_5 \neq 0 \). Then

\[ -\beta \sin \beta L = 0 \]
\[ \sin \beta L = 0 \]
\[ \beta L = n\pi, \quad n = 1, 2, \ldots \]
\[ \beta_n = \frac{n\pi}{L}. \]

There are negative eigenvalues \( \lambda = -n^2\pi^2/L^2 \), and the eigenfunctions associated with them are

\[ X(x) = C_5 \cos \beta x + C_6 \sin \beta x \]
\[ = C_5 \cos \beta x \quad \rightarrow \quad X_n(x) = \cos \frac{n\pi x}{L}. \]
$n$ only takes on the values it does because negative integers result in redundant values for $\lambda$. With this formula for $\lambda$, the ODE for $T$ becomes

$$\frac{1}{kT} \frac{dT}{dt} = -\frac{n^2 \pi^2}{L^2}.$$  

Multiply both sides by $kT$.

$$\frac{dT}{dt} = -\frac{kn^2 \pi^2}{L^2} T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp \left( -\frac{kn^2 \pi^2}{L^2} t \right) \rightarrow T_n(t) = \exp \left( -\frac{kn^2 \pi^2}{L^2} t \right)$$

According to the principle of superposition, the general solution to the PDE for $u$ is a linear combination of $X_n(x) T_n(t)$ over all the eigenvalues.

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \exp \left( -\frac{kn^2 \pi^2}{L^2} t \right) \cos \frac{n\pi x}{L}$$

**Part (c)**

Apply the initial condition $u(x, 0) = f(x)$ to determine $A_0$ and $A_n$.

$$u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x) \quad (1)$$

**Part (d)**

To find $A_0$, integrate both sides of equation (1) with respect to $x$ from 0 to $x$.

$$\int_0^L \left( A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \right) dx = \int_0^L f(x) dx$$

Split up the integral on the left into two and bring the constants in front.

$$A_0 \int_0^L dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} dx = \int_0^L f(x) dx$$

Consequently,

$$A_0 L = \int_0^L f(x) dx$$

$$A_0 = \frac{1}{L} \int_0^L f(x) dx.$$

To find $A_n$, multiply both sides of equation (1) by $\cos(m\pi x/L)$, where $m$ is an integer,

$$A_0 \cos \frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} = f(x) \cos \frac{m\pi x}{L}$$
and then integrate both sides with respect to $x$ from 0 to $L$.

$$\int_0^L \left( A_0 \cos \frac{m \pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n \pi x}{L} \cos \frac{m \pi x}{L} \right) \, dx = \int_0^L f(x) \cos \frac{m \pi x}{L} \, dx$$

Split up the integral on the left into two and bring the constants in front.

$$A_0 \int_0^L \cos \frac{m \pi x}{L} \, dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n \pi x}{L} \cos \frac{m \pi x}{L} \, dx = \int_0^L f(x) \cos \frac{m \pi x}{L} \, dx$$

Because the cosine functions are orthogonal, the remaining integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$A_n \int_0^L \cos^2 \frac{n \pi x}{L} \, dx = \int_0^L f(x) \cos \frac{n \pi x}{L} \, dx$$

Consequently,

$$A_n \left( \frac{L}{2} \right) = \int_0^L f(x) \cos \frac{n \pi x}{L} \, dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n \pi x}{L} \, dx.$$ 

**Part (e)**

The temperature distribution approaches equilibrium as $t \to \infty$.

$$\lim_{t \to \infty} u(x, t) = \lim_{t \to \infty} \left[ A_0 + \sum_{n=1}^{\infty} A_n \exp \left( -\frac{k n^2 \pi^2}{L^2} t \right) \cos \frac{n \pi x}{L} \right]$$

$$= A_0$$

$$= \frac{1}{L} \int_0^L f(x) \, dx$$

In particular, the equilibrium temperature distribution is the average of the initial temperature distribution.