Exercise 2.3.8

Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u.$$

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° ($\alpha > 0$, see Exercise 1.2.4) or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

(b) Solve the time-dependent problem $[u(x,0) = f(x)]$ if $\alpha > 0$. Analyze the temperature for large time ($t \to \infty$) and compare to part (a).

Solution

Part (a)

The equilibrium temperature distributions have no time dependence: $u_E = u_E(x)$. As a result, they satisfy

$$0 = k\frac{d^2 u_E}{dx^2} - \alpha u_E.$$

Divide both sides by k.

$$\frac{d^2 u_E}{dx^2} - \frac{\alpha}{k}u_E = 0$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$u_E(x) = C_1 \cosh\sqrt{\frac{\alpha}{k}}x + C_2 \sinh\sqrt{\frac{\alpha}{k}}x$$

Since the boundary conditions for u apply for all time, u_E satisfies the same conditions, $u_E(0) = 0$ and $u_E(L) = 0$. Apply them both to determine C_1 and C_2.

$$\begin{aligned} u_E(0) &= C_1 = 0 \\ u_E(L) &= C_1 \cosh\sqrt{\frac{\alpha}{k}}L + C_2 \sinh\sqrt{\frac{\alpha}{k}}L = 0 \end{aligned}$$

The second equation reduces to $C_2 \sinh\sqrt{\frac{\alpha}{k}}L = 0$. The only way this equation is satisfied is if $C_2 = 0$, which means the only equilibrium temperature distribution is

$$u_E(x) = 0.$$

Part (b)

The PDE and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u \quad \rightarrow \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)] - \alpha[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned} u(0,t) &= 0 \quad &\rightarrow \quad X(0)T(t) &= 0 \quad &\rightarrow \quad X(0) &= 0 \\ u(L,t) &= 0 \quad &\rightarrow \quad X(L)T(t) &= 0 \quad &\rightarrow \quad X(L) &= 0 \end{aligned}$$

Separate variables in the PDE now.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2} - \alpha X(x)T(t)$$

Divide both sides by $kX(x)T(t)$.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2} - \frac{\alpha}{k}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} - \frac{\alpha}{k} = \lambda$$

As a result of using the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} &= \lambda \\ \frac{1}{X}\frac{d^2X}{dx^2} - \frac{\alpha}{k} &= \lambda \end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \mu^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \left(\frac{\alpha}{k} + \mu^2\right)X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh\sqrt{\frac{\alpha}{k} + \mu^2}x + C_4 \sinh\sqrt{\frac{\alpha}{k} + \mu^2}x$$

Apply the boundary conditions now to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_3 = 0 \\ X(L) &= C_3 \cosh\sqrt{\frac{\alpha}{k} + \mu^2}L + C_4 \sinh\sqrt{\frac{\alpha}{k} + \mu^2}L = 0 \end{aligned}$$

The second equation reduces to $C_4 \sinh \sqrt{\frac{\alpha}{k} + \mu^2} L = 0$. Since hyperbolic sine is not oscillatory, the only way this equation is satisfied is if $C_4 = 0$. The trivial solution $X(x) = 0$ results, which means there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \frac{\alpha}{k} X,$$

which is the same as the one for $u_E(x)$. Since the boundary conditions are the same, the trivial solution $X(x) = 0$ is obtained, so zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\gamma^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \left(\frac{\alpha}{k} - \gamma^2\right) X.$$

It was found earlier that the solution with hyperbolic sine and hyperbolic cosine led to the trivial solution. The same will happen here unless the quantity in parentheses is negative.

$$\frac{d^2X}{dx^2} = -\left(\gamma^2 - \frac{\alpha}{k}\right) X$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \sqrt{\gamma^2 - \frac{\alpha}{k}} x + C_6 \sin \sqrt{\gamma^2 - \frac{\alpha}{k}} x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(L) &= C_5 \cos \sqrt{\gamma^2 - \frac{\alpha}{k}} L + C_6 \sin \sqrt{\gamma^2 - \frac{\alpha}{k}} L = 0 \end{aligned}$$

The second equation reduces to $C_6 \sin \sqrt{\gamma^2 - \frac{\alpha}{k}} L = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin \sqrt{\gamma^2 - \frac{\alpha}{k}} L &= 0 \\ \sqrt{\gamma^2 - \frac{\alpha}{k}} L &= n\pi, \quad n = 1, 2, \ldots \\ \sqrt{\gamma^2 - \frac{\alpha}{k}} &= \frac{n\pi}{L} \\ \gamma^2 - \frac{\alpha}{k} &= \frac{n^2\pi^2}{L^2} \\ \gamma^2 &= \frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}. \end{aligned}$$

The negative eigenvalues are $\lambda = -\alpha/k - n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos \sqrt{\gamma^2 - \frac{\alpha}{k}} x + C_6 \sin \sqrt{\gamma^2 - \frac{\alpha}{k}} x \\ &= C_6 \sin \sqrt{\gamma^2 - \frac{\alpha}{k}} x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{L}. \end{aligned}$$

n only takes on the values it does because negative integers result in redundant values for λ. Now solve the ODE for T with this formula for λ.

$$\begin{aligned}\frac{dT}{dt} &= k\left(-\frac{\alpha}{k} - \frac{n^2\pi^2}{L^2}\right)T \\ &= -k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)T\end{aligned}$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left[-k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)t\right] \quad \rightarrow \quad T_n(t) = \exp\left[-k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)t\right]$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ over all the eigenvalues.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k} + \frac{n^2\pi^2}{L^2}\right)t\right]\sin\frac{n\pi x}{L}}$$

Apply the initial condition $u(x,0) = f(x)$ to determine A_n.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$$

Multiply both sides by $\sin(m\pi x/L)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = f(x)\sin\frac{m\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$B_n \int_0^L \sin^2\frac{n\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Evaluate the integral on the left.

$$B_n\left(\frac{L}{2}\right) = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

So then

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.}$$

Take the limit of $u(x,t)$ as $t \to \infty$ to find the equilibrium temperature distribution.

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\sum_{n=1}^{\infty} B_n \exp\left[-k\left(\frac{\alpha}{k}+\frac{n^2\pi^2}{L^2}\right)t\right]\sin\frac{n\pi x}{L}\\ &= 0\end{aligned}$$

This result agrees with the one from part (a).