Exercise 2.4.1

Solve the heat equation \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \), \( 0 < x < L \), \( t > 0 \), subject to

\[
\begin{align*}
\frac{\partial u}{\partial x}(0, t) &= 0 \quad t > 0 \\
\frac{\partial u}{\partial x}(L, t) &= 0 \quad t > 0.
\end{align*}
\]

(a) \( u(x, 0) = \begin{cases} 
0 & x < L/2 \\
1 & x > L/2
\end{cases} \)

(b) \( u(x, 0) = 6 + 4 \cos \frac{3\pi x}{L} \)

(c) \( u(x, 0) = -2 \sin \frac{\pi x}{L} \)

(d) \( u(x, 0) = -3 \cos \frac{8\pi x}{L} \)

Solution

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form \( u(x, t) = X(x)T(t) \) and substitute it into the PDE

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad \frac{\partial}{\partial t} [X(x)T(t)] = k \frac{\partial^2}{\partial x^2} [X(x)T(t)]
\]

and the boundary conditions.

\[
\begin{align*}
\frac{\partial u}{\partial x}(0, t) &= 0 \quad \Rightarrow \quad X'(0)T(t) = 0 \quad \Rightarrow \quad X'(0) = 0 \\
\frac{\partial u}{\partial x}(L, t) &= 0 \quad \Rightarrow \quad X'(L)T(t) = 0 \quad \Rightarrow \quad X'(L) = 0
\end{align*}
\]

Now separate variables in the PDE.

\[
X \frac{dT}{dt} = kT \frac{d^2X}{dx^2}
\]

Divide both sides by \( kX(x)T(t) \). Note that the final answer for \( u \) will be the same regardless which side \( k \) is on. Constants are normally grouped with \( t \).

\[
\frac{1}{kT} \frac{dT}{dt} = \frac{1}{X} \frac{d^2X}{dx^2}
\]

The only way a function of \( t \) can be equal to a function of \( x \) is if both are equal to a constant \( \lambda \).

\[
\frac{1}{kT} \frac{dT}{dt} = \frac{1}{X} \frac{d^2X}{dx^2} = \lambda
\]

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in \( x \) and one in \( t \).

\[
\begin{align*}
\frac{1}{kT} \frac{dT}{dt} &= \lambda \\
\frac{1}{X} \frac{d^2X}{dx^2} &= \lambda
\end{align*}
\]

www.stemjock.com
Values of $\lambda$ that result in nontrivial solutions for $X$ and $T$ are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that $\lambda$ is positive: $\lambda = \alpha^2$. The ODE for $X$ becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$  

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative with respect to $x$.

$$X'(x) = \alpha (C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions now to determine $C_1$ and $C_2$.

$$X'(0) = \alpha (C_2) = 0$$
$$X'(L) = \alpha (C_1 \sinh \alpha L + C_2 \cosh \alpha L) = 0$$

The first equation implies that $C_2 = 0$, so the second equation reduces to $C_1 \alpha \sinh \alpha L = 0$. Because hyperbolic sine is not oscillatory, $C_1$ must be zero for the equation to be satisfied. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that $\lambda$ is zero: $\lambda = 0$. The ODE for $X$ becomes

$$\frac{d^2X}{dx^2} = 0.$$  

The general solution is obtained by integrating both sides with respect to $x$ twice.

$$\frac{dX}{dx} = C_3$$

Apply the boundary conditions now.

$$X'(0) = C_3 = 0$$
$$X'(L) = C_3 = 0$$

Consequently,

$$\frac{dX}{dx} = 0.$$  

Integrate both sides with respect to $x$ once more.

$$X(x) = C_4$$

Zero is an eigenvalue because $X(x)$ is not zero. The eigenfunction associated with it is $X_0(x) = 1$. Solve the ODE for $T$ now with $\lambda = 0$.

$$\frac{dT}{dt} = 0 \quad \rightarrow \quad T_0(t) = \text{constant}$$

Suppose thirdly that $\lambda$ is negative: $\lambda = -\beta^2$. The ODE for $X$ becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$
The general solution is written in terms of sine and cosine.

\[ X(x) = C_5 \cos \beta x + C_6 \sin \beta x \]

Take a derivative of it with respect to \( x \).

\[ X'(x) = \beta (-C_5 \sin \beta x + C_6 \cos \beta x) \]

Apply the boundary conditions now to determine \( C_5 \) and \( C_6 \).

\[ X'(0) = \beta (C_6) = 0 \]
\[ X'(L) = \beta (-C_5 \sin \beta L + C_6 \cos \beta L) = 0 \]

The first equation implies that \( C_6 = 0 \), so the second equation reduces to \(-C_5 \beta \sin \beta L = 0\). To avoid the trivial solution, we insist that \( C_5 \neq 0 \). Then

\[ -\beta \sin \beta L = 0 \]
\[ \sin \beta L = 0 \]
\[ \beta L = n\pi, \quad n = 1, 2, \ldots \]
\[ \beta_n = \frac{n\pi}{L} \]

There are negative eigenvalues \( \lambda = -n^2\pi^2/L^2 \), and the eigenfunctions associated with them are

\[ X(x) = C_5 \cos \beta x + C_6 \sin \beta x \]
\[ = C_5 \cos \beta x \quad \rightarrow \quad X_n(x) = \cos \frac{n\pi x}{L}. \]

\( n \) only takes on the values it does because negative integers result in redundant values for \( \lambda \). With this formula for \( \lambda \), the ODE for \( T \) becomes

\[ \frac{1}{kT} \frac{dT}{dt} = -\frac{n^2\pi^2}{L^2}. \]

Multiply both sides by \( kT \).

\[ \frac{dT}{dt} = -\frac{k n^2 \pi^2}{L^2} T \]

The general solution is written in terms of the exponential function.

\[ T(t) = C_7 \exp \left( -\frac{k n^2 \pi^2}{L^2} t \right) \quad \rightarrow \quad T_n(t) = \exp \left( -\frac{k n^2 \pi^2}{L^2} t \right) \]

According to the principle of superposition, the general solution to the PDE for \( u \) is a linear combination of \( X_n(x)T_n(t) \) over all the eigenvalues.

\[ u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \exp \left( -\frac{k n^2 \pi^2}{L^2} t \right) \cos \frac{n\pi x}{L} \]

Use the initial condition \( u(x, 0) = f(x) \) to determine \( A_0 \) and \( A_n \).

\[ u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x) \]

www.stemjock.com
Part (a)

Here \( f(x) = 0 \) for \( x < L/2 \) and \( f(x) = 1 \) for \( x > L/2 \).

\[
    u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x) \tag{1}
\]

To find \( A_0 \), integrate both sides of equation (1) with respect to \( x \) from 0 to \( L \).

\[
    \int_0^L \left( A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \right) \, dx = \int_0^L f(x) \, dx
\]

Split up the integral on the left into two and bring the constants in front. Write out the integral on the right.

\[
    A_0 \int_0^L \, dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \, dx = \int_0^{L/2} (0) \, dx + \int_{L/2}^L (1) \, dx
\]

Evaluate the integrals.

\[
    A_0 L = \frac{L}{2}
\]

\[
    A_0 = \frac{1}{2}
\]

To find \( A_n \), multiply both sides of equation (1) by \( \cos(m\pi x/L) \), where \( m \) is a positive integer,

\[
    A_0 \cos \frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} = f(x) \cos \frac{m\pi x}{L}
\]

and then integrate both sides with respect to \( x \) from 0 to \( L \).

\[
    \int_0^L \left( A_0 \cos \frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \right) \, dx = \int_0^L f(x) \cos \frac{m\pi x}{L} \, dx
\]

Split up the integral on the left into two and bring the constants in front. Write out the integral on the right.

\[
    A_0 \int_0^L \cos \frac{m\pi x}{L} \, dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = \int_0^{L/2} (0) \cos \frac{m\pi x}{L} \, dx + \int_{L/2}^L (1) \cos \frac{m\pi x}{L} \, dx
\]

Because the cosine functions are orthogonal, the second integral on the left is zero if \( n \neq m \). As a result, every term in the infinite series vanishes except for the \( n = m \) one.

\[
    A_n \int_0^L \cos^2 \frac{n\pi x}{L} \, dx = \int_{L/2}^L \cos \frac{n\pi x}{L} \, dx
\]

Evaluate the integrals.

\[
    A_n \left( \frac{L}{2} \right) = -\frac{L}{n\pi} \sin \frac{n\pi}{2}
\]

www.stemjock.com
\[ A_n = -\frac{2}{n\pi} \sin \frac{n\pi}{2} \]

The general solution then becomes

\[
u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left( -\frac{2}{n\pi} \sin \frac{n\pi}{2} \right) \exp \left( -\frac{kn^2\pi^2}{L^2} t \right) \cos \frac{n\pi x}{L} \]

\[
= \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \exp \left( -\frac{kn^2\pi^2}{L^2} t \right) \cos \frac{n\pi x}{L}.
\]

Notice that the summand is zero for even values of \( n \). The answer can thus be simplified (that is, made to converge faster) by summing over the odd integers only. Make the substitution \( n = 2p - 1 \) in the sum.

\[
u(x,t) = \frac{1}{2} - \frac{2}{\pi} \sum_{2p-1=1}^{\infty} \frac{\sin \frac{(2p-1)\pi}{2}}{2p-1} \exp \left( -\frac{k(2p-1)^2\pi^2}{L^2} t \right) \cos \frac{(2p-1)\pi x}{L}.
\]

Therefore,

\[
u(x,t) = \frac{1}{2} + \frac{2}{\pi} \sum_{p=1}^{\infty} (-1)^p \exp \left( -\frac{k(2p-1)^2\pi^2}{L^2} t \right) \cos \frac{(2p-1)\pi x}{L}.
\]

**Part (b)**

Here \( f(x) = 6 + 4 \cos \frac{3\pi x}{L} \).

\[
u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = 6 + 4 \cos \frac{3\pi x}{L}
\]

By inspection we see that the coefficients are

\[ A_0 = 6 \]

\[ A_n = \begin{cases} 0 & \text{if } n \neq 3 \\ 4 & \text{if } n = 3 \end{cases} \]

Therefore,

\[
u(x,t) = 6 + 4 \exp \left( -\frac{9\pi^2k}{L^2} t \right) \cos \frac{3\pi x}{L}.
\]

**Part (c)**

Here \( f(x) = -2 \sin \frac{\pi x}{L} \).

\[
u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = -2 \sin \frac{\pi x}{L}
\]

(2)

To find \( A_0 \), integrate both sides of equation (2) with respect to \( x \) from 0 to \( L \).

\[
\int_0^L \left( A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \right) dx = -\int_0^L 2 \sin \frac{\pi x}{L} dx
\]

www.stemjock.com
Split up the integral on the left into two and bring the constants in front.

\[ A_0 \int_0^L dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} dx = - \int_0^L 2 \sin \frac{\pi x}{L} dx \]

Evaluate the integrals.

\[ A_0 L = -\frac{4L}{\pi} \]
\[ A_0 = -\frac{4}{\pi} \]

To find \( A_n \), multiply both sides of equation (2) by \( \cos(\frac{m\pi x}{L}) \), where \( m \) is a positive integer,

\[ A_0 \cos \frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} = -2 \sin \frac{\pi x}{L} \cos \frac{m\pi x}{L} \]

and then integrate both sides with respect to \( x \) from 0 to \( L \).

\[ \int_0^L \left( A_0 \cos \frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \right) dx = - \int_0^L 2 \sin \frac{\pi x}{L} \cos \frac{m\pi x}{L} dx \]

Split up the integral on the left into two and bring the constants in front.

\[ A_0 \int_0^L \cos \frac{m\pi x}{L} dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = - \int_0^L 2 \sin \frac{\pi x}{L} \cos \frac{m\pi x}{L} dx \]

Because the cosine functions are orthogonal, the second integral on the left is zero if \( n \neq m \). As a result, every term in the infinite series vanishes except for the \( n = m \) one.

\[ A_n \int_0^L \cos^2 \frac{n\pi x}{L} dx = - \int_0^L 2 \sin \frac{\pi x}{L} \cos \frac{n\pi x}{L} dx \]

\[ A_n \left( \frac{L}{2} \right) = \begin{cases} 
0 & \text{if } n = 1 \\
\frac{2L}{\pi} \frac{1 + (-1)^n}{n^2 - 1} & \text{if } n \neq 1 
\end{cases} \]

\[ A_n = \begin{cases} 
0 & \text{if } n = 1 \\
\frac{4}{\pi} \frac{1 + (-1)^n}{n^2 - 1} & \text{if } n \neq 1 
\end{cases} \]

The general solution then becomes

\[ u(x, t) = -\frac{4}{\pi} + \sum_{n=2}^{\infty} \left[ \frac{4}{\pi} \frac{1 + (-1)^n}{n^2 - 1} \right] \exp \left( -\frac{kn^2\pi^2}{L^2} t \right) \cos \frac{n\pi x}{L} \]

Notice that the summand is zero if \( n \) is odd. The solution can thus be simplified (that is, made to converge faster) by summing over the even integers only. Make the substitution \( n = 2p \) in the sum.

\[ u(x, t) = -\frac{4}{\pi} + \sum_{2p=2}^{\infty} \left[ \frac{4}{\pi} \frac{2}{(2p)^2 - 1} \right] \exp \left( -\frac{k(2p)^2\pi^2}{L^2} t \right) \cos \frac{2p\pi x}{L} \]

www.stemjock.com
Therefore,
\[ u(x, t) = \frac{-4}{\pi} + \frac{8}{\pi} \sum_{p=1}^{\infty} \frac{1}{4p^2 - 1} \exp \left( -\frac{4\pi^2 p^2 k}{L^2} t \right) \cos \frac{2p\pi x}{L}. \]

Part (d)

Here \( f(x) = -3 \cos \frac{8\pi x}{L} \).

Therefore,
\[ u(x, t) = -3 \exp \left( -\frac{64\pi^2 k}{L^2} t \right) \cos \frac{8\pi x}{L}. \]