Exercise 2.4.5

This problem presents an alternative derivation of the heat equation for a thin wire. The equation for a circular wire of finite thickness is the two-dimensional heat equation (in polar coordinates). Show that this reduces to (2.4.25) if the temperature does not depend on \( r \) and if the wire is very thin.

Solution

The two-dimensional heat equation in polar coordinates is

\[
\frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right).
\]

If the temperature \( u \) does not depend on \( r \), then the radial derivative vanishes.

\[
\frac{\partial u}{\partial t} = k \left( \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)
\]

For a very thin wire that is bent into a circle with radius \( R \), \( r = R \).

\[
\frac{\partial u}{\partial t} = k \left( \frac{1}{R^2} \frac{\partial^2 u}{\partial \theta^2} \right)
\]

\[
= k \left( \frac{\partial^2 u}{\partial (R\theta)^2} \right)
\]

Letting \( x = R\theta \) represent the arc length, we obtain equation (2.4.25) in the text.

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}
\]

(2.4.25)