Exercise 2.4.5

This problem presents an alternative derivation of the heat equation for a thin wire. The equation for a circular wire of finite thickness is the two-dimensional heat equation (in polar coordinates). Show that this reduces to (2.4.25) if the temperature does not depend on r and if the wire is very thin.

Solution

The two-dimensional heat equation in polar coordinates is

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right).$$

If the temperature u does not depend on r, then the radial derivative vanishes.

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

For a very thin wire that is bent into a circle with radius R, $r = R$.

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\left(\frac{1}{R^2}\frac{\partial^2 u}{\partial \theta^2}\right)\\ &= k\left(\frac{\partial^2 u}{\partial (R\theta)^2}\right)\end{aligned}$$

Letting $x = R\theta$ represent the arc length, we obtain equation (2.4.25) in the text.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{2.4.25}$$