Exercise 2.4.7

Solve the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

[*Hints*: It is known that if $u(x,t) = \phi(x)G(t)$, then $\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2}$. Appropriately assume $\lambda > 0$. Assume the eigenfunctions $\phi_n(x)$ satisfy the following integral condition (orthogonality):

$$\int_0^L \phi_n(x)\phi_m(x)\,dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases}$$

subject to the following conditions:

(a) $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = f(x)$

(b) $u(0,t) = 0$, $\frac{\partial u}{\partial \mathrm{x}}(L,t) = 0$, $u(x,0) = f(x)$

(c) $\frac{\partial u}{\partial x}(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = f(x)$

(d) $\frac{\partial u}{\partial \mathrm{x}}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$, $u(x,0) = f(x)$ and modify orthogonal condition [using Table 2.4.1.]

[TYPO: These x's should be italicized. Also, the square bracket is not paired.]

Solution

Part (a)

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 \\
&u(0,t) = 0 \\
&u(L,t) = 0 \\
&u(x,0) = f(x)
\end{aligned}$$

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned}
u(0,t) = 0 \quad &\rightarrow \quad X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0 \\
u(L,t) = 0 \quad &\rightarrow \quad X(L)T(t) = 0 \quad \rightarrow \quad X(L) = 0
\end{aligned}$$

Now separate variables in the PDE.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. Note that the final answer for u will be the same regardless which side k is on. Constants are normally grouped with t.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} &= \lambda \\ \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(L) &= C_1 \cosh \alpha L + C_2 \sinh \alpha L = 0 \end{aligned}$$

The second equation reduces to $C_2 \sinh \alpha L = 0$. Because hyperbolic sine is not oscillatory, C_2 must be zero for the equation to be satisfied. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions now to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(L) &= C_3 L + C_4 = 0 \end{aligned}$$

The second equation reduces to $C_3 = 0$. This results in the trivial solution $X(x) = 0$, which means zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(L) &= C_5 \cos \beta L + C_6 \sin \beta L = 0 \end{aligned}$$

The second equation reduces to $C_6 \sin \beta L = 0$. To avoid the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

There are negative eigenvalues $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos \beta x + C_6 \sin \beta x \\ &= C_6 \sin \beta x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{L}. \end{aligned}$$

n only takes on the values it does because negative integers result in redundant values for λ. With this formula for λ, the ODE for T becomes

$$\frac{1}{kT}\frac{dT}{dt} = -\frac{n^2\pi^2}{L^2}.$$

Multiply both sides by kT.

$$\frac{dT}{dt} = -\frac{kn^2\pi^2}{L^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \quad \rightarrow \quad T_n(t) = \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ for each of the eigenvalues.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \sin \frac{n\pi x}{L}}$$

Apply the initial condition $u(x, 0) = f(x)$ now to determine B_n.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x)$$

Multiply both sides by $\sin(m\pi x/L)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} = f(x) \sin \frac{m\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the one where $n = m$.

$$B_n \int_0^L \sin^2\frac{n\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Evaluate the integral on the left.

$$B_n\left(\frac{L}{2}\right) = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

So then

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.}$$

Part (b)

$$\begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 \\ &u(0,t) = 0 \\ &\frac{\partial u}{\partial x}(L,t) = 0 \\ &u(x,0) = f(x) \end{aligned}$$

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned} u(0,t) &= 0 & &\rightarrow & X(0)T(t) &= 0 & &\rightarrow & X(0) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 & &\rightarrow & X'(L)T(t) &= 0 & &\rightarrow & X'(L) &= 0 \end{aligned}$$

Now separate variables in the PDE.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. Note that the final answer for u will be the same regardless which side k is on. Constants are normally grouped with t.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} &= \lambda \\ \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative with respect to x.

$$X'(x) = \alpha(C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X'(L) &= \alpha(C_1 \sinh \alpha L + C_2 \cosh \alpha L) = 0 \end{aligned}$$

The second equation reduces to $C_2 \alpha \cosh \alpha L = 0$. Because no nonzero value of α satisfies this equation, C_2 must be zero. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{dX}{dx} = C_3$$

Apply the boundary conditions at $x = L$ now.

$$X'(L) = C_3 = 0$$

Consequently,

$$\frac{dX}{dx} = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_4$$

Apply the boundary conditions at $x = 0$ now.

$$X(0) = C_4 = 0$$

The trivial solution $X(x) = 0$ is obtained, so zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Take a derivative of it with respect to x.

$$X'(x) = \beta(-C_5 \sin \beta x + C_6 \cos \beta x)$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X'(L) &= \beta(-C_5 \sin \beta L + C_6 \cos \beta L) = 0 \end{aligned}$$

The second equation reduces to $C_6\beta\cos\beta L = 0$. To avoid the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned}\cos\beta L &= 0\\ \beta L &= \frac{1}{2}(2n-1)\pi, \quad n = 1, 2, \ldots\\ \beta_n &= \frac{1}{2L}(2n-1)\pi.\end{aligned}$$

There are negative eigenvalues $\lambda = -(2n-1)^2\pi^2/4L^2$, and the eigenfunctions associated with them are

$$\begin{aligned}X(x) &= C_5\cos\beta x + C_6\sin\beta x\\ &= C_6\sin\beta x \quad \rightarrow \quad X_n(x) = \sin\frac{(2n-1)\pi x}{2L}.\end{aligned}$$

n only takes on the values it does because negative integers result in redundant values for λ. With this formula for λ, the ODE for T becomes

$$\frac{1}{kT}\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2}{4L^2}.$$

Multiply both sides by kT.

$$\frac{dT}{dt} = -k\frac{(2n-1)^2\pi^2}{4L^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7\exp\left(-k\frac{(2n-1)^2\pi^2}{4L^2}t\right) \quad \rightarrow \quad T_n(t) = \exp\left(-\frac{k(2n-1)^2\pi^2}{4L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ over all the eigenvalues.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n\exp\left(-\frac{k(2n-1)^2\pi^2}{4L^2}t\right)\sin\frac{(2n-1)\pi x}{2L}}$$

Now use the initial condition $u(x,0) = f(x)$ to determine A_n.

$$u(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{(2n-1)\pi x}{2L} = f(x)$$

Multiply both sides by $\sin[(2m-1)\pi x/2L]$, where m is an integer,

$$\sum_{n=1}^{\infty} B_n\sin\frac{(2n-1)\pi x}{2L}\sin\frac{(2m-1)\pi x}{2L} = f(x)\sin\frac{(2m-1)\pi x}{2L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} B_n\sin\frac{(2n-1)\pi x}{2L}\sin\frac{(2m-1)\pi x}{2L}\,dx = \int_0^L f(x)\sin\frac{(2m-1)\pi x}{2L}\,dx$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{(2n-1)\pi x}{2L} \sin\frac{(2m-1)\pi x}{2L}\, dx = \int_0^L f(x) \sin\frac{(2m-1)\pi x}{2L}\, dx$$

The sine functions are orthogonal, so the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$B_n \int_0^L \sin^2\frac{(2n-1)\pi x}{2L}\, dx = \int_0^L f(x) \sin\frac{(2n-1)\pi x}{2L}\, dx$$

Evaluate the integral on the left side.

$$B_n \left(\frac{L}{2}\right) = \int_0^L f(x) \sin\frac{(2n-1)\pi x}{2L}\, dx$$

Therefore,

$$\boxed{B_n = \frac{2}{L} \int_0^L f(x) \sin\frac{(2n-1)\pi x}{2L}\, dx.}$$

Part (c)

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 \\
&\frac{\partial u}{\partial x}(0,t) = 0 \\
&u(L,t) = 0 \\
&u(x,0) = f(x)
\end{aligned}$$

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned}
\frac{\partial u}{\partial x}(0,t) = 0 \quad &\rightarrow \quad X'(0)T(t) = 0 \quad &\rightarrow \quad X'(0) = 0 \\
u(L,t) = 0 \quad &\rightarrow \quad X(L)T(t) = 0 \quad &\rightarrow \quad X(L) = 0
\end{aligned}$$

Now separate variables in the PDE.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. Note that the final answer for u will be the same regardless which side k is on. Constants are normally grouped with t.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned}
\frac{1}{kT}\frac{dT}{dt} &= \lambda \\
\frac{1}{X}\frac{d^2X}{dx^2} &= \lambda
\end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative with respect to x.

$$X'(x) = \alpha(C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= \alpha(C_2) = 0 \\ X(L) &= C_1 \cosh \alpha L + C_2 \sinh \alpha L = 0 \end{aligned}$$

The first equation implies that $C_2 = 0$, so the second equation reduces to $C_1 \cosh \alpha L = 0$. Because hyperbolic cosine is not oscillatory, C_1 must be zero for the equation to be satisfied. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{dX}{dx} = C_3$$

Apply the boundary conditions at $x = 0$ now.

$$X'(0) = C_3 = 0$$

Consequently,

$$\frac{dX}{dx} = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_4$$

Apply the boundary conditions at $x = L$ now.

$$X(L) = C_4 = 0$$

The trivial solution $X(x) = 0$ is obtained, so zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Take a derivative of it with respect to x.

$$X'(x) = \beta(-C_5 \sin \beta x + C_6 \cos \beta x)$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} X'(0) &= \beta(C_6) = 0 \\ X(L) &= C_5 \cos \beta L + C_6 \sin \beta L = 0 \end{aligned}$$

The first equation implies that $C_6 = 0$, so the second equation reduces to $C_5 \cos\beta L = 0$. To avoid the trivial solution, we insist that $C_5 \neq 0$. Then

$$\begin{aligned}\cos\beta L &= 0\\ \beta L &= \frac{1}{2}(2n-1)\pi, \quad n = 1, 2, \ldots\\ \beta_n &= \frac{1}{2L}(2n-1)\pi.\end{aligned}$$

There are negative eigenvalues $\lambda = -(2n-1)^2\pi^2/4L^2$, and the eigenfunctions associated with them are

$$\begin{aligned}X(x) &= C_5 \cos\beta x + C_6 \sin\beta x\\ &= C_5 \cos\beta x \quad \rightarrow \quad X_n(x) = \cos\frac{(2n-1)\pi x}{2L}.\end{aligned}$$

n only takes on the values it does because negative integers result in redundant values for λ. With this formula for λ, the ODE for T becomes

$$\frac{1}{kT}\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2}{4L^2}.$$

Multiply both sides by kT.

$$\frac{dT}{dt} = -k\frac{(2n-1)^2\pi^2}{4L^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left(-k\frac{(2n-1)^2\pi^2}{4L^2}t\right) \quad \rightarrow \quad T_n(t) = \exp\left(-\frac{k(2n-1)^2\pi^2}{4L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ over all the eigenvalues.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{k(2n-1)^2\pi^2}{4L^2}t\right)\cos\frac{(2n-1)\pi x}{2L}}$$

Now use the initial condition $u(x, 0) = f(x)$ to determine A_n.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi x}{2L} = f(x)$$

Multiply both sides by $\cos[(2m-1)\pi x/2L]$, where m is an integer,

$$\sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi x}{2L}\cos\frac{(2m-1)\pi x}{2L} = f(x)\cos\frac{(2m-1)\pi x}{2L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi x}{2L}\cos\frac{(2m-1)\pi x}{2L}\,dx = \int_0^L f(x)\cos\frac{(2m-1)\pi x}{2L}\,dx$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{(2n-1)\pi x}{2L} \cos\frac{(2m-1)\pi x}{2L}\,dx = \int_0^L f(x)\cos\frac{(2m-1)\pi x}{2L}\,dx$$

The cosine functions are orthogonal, so the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$A_n \int_0^L \cos^2\frac{(2n-1)\pi x}{2L} = \int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

Evaluate the integral on the left side.

$$A_n\left(\frac{L}{2}\right) = \int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

Therefore,

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.}$$

Part (d)

$$\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 \\
\frac{\partial u}{\partial x}(0,t) &= 0 \\
\frac{\partial u}{\partial x}(L,t) &= 0 \\
u(x,0) &= f(x)
\end{aligned}$$

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \to \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned}
\frac{\partial u}{\partial x}(0,t) = 0 \quad &\to \quad X'(0)T(t) = 0 \quad \to \quad X'(0) = 0 \\
\frac{\partial u}{\partial x}(L,t) = 0 \quad &\to \quad X'(L)T(t) = 0 \quad \to \quad X'(L) = 0
\end{aligned}$$

Now separate variables in the PDE.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. Note that the final answer for u will be the same regardless which side k is on. Constants are normally grouped with t.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned}
\frac{1}{kT}\frac{dT}{dt} &= \lambda \\
\frac{1}{X}\frac{d^2X}{dx^2} &= \lambda
\end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative with respect to x.

$$X'(x) = \alpha(C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= \alpha(C_2) = 0 \\ X'(L) &= \alpha(C_1 \sinh \alpha L + C_2 \cosh \alpha L) = 0 \end{aligned}$$

The first equation implies that $C_2 = 0$, so the second equation reduces to $C_1 \alpha \sinh \alpha L = 0$. Because hyperbolic sine is not oscillatory, C_1 must be zero for the equation to be satisfied. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{dX}{dx} = C_3$$

Apply the boundary conditions now.

$$\begin{aligned} X'(0) &= C_3 = 0 \\ X'(L) &= C_3 = 0 \end{aligned}$$

Consequently,

$$\frac{dX}{dx} = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_4$$

Zero is an eigenvalue because $X(x)$ is not zero. The eigenfunction associated with it is $X_0(x) = 1$. Solve the ODE for T now with $\lambda = 0$.

$$\frac{dT}{dt} = 0 \quad \rightarrow \quad T_0(t) = \text{constant}$$

Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Take a derivative of it with respect to x.

$$X'(x) = \beta(-C_5 \sin \beta x + C_6 \cos \beta x)$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} X'(0) &= \beta(C_6) = 0 \\ X'(L) &= \beta(-C_5 \sin \beta L + C_6 \cos \beta L) = 0 \end{aligned}$$

The first equation implies that $C_6 = 0$, so the second equation reduces to $-C_5\beta \sin \beta L = 0$. To avoid the trivial solution, we insist that $C_5 \neq 0$. Then

$$\begin{aligned} -\beta \sin \beta L &= 0 \\ \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

There are negative eigenvalues $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos \beta x + C_6 \sin \beta x \\ &= C_5 \cos \beta x \quad \rightarrow \quad X_n(x) = \cos \frac{n\pi x}{L}. \end{aligned}$$

n only takes on the values it does because negative integers result in redundant values for λ. With this formula for λ, the ODE for T becomes

$$\frac{1}{kT}\frac{dT}{dt} = -\frac{n^2\pi^2}{L^2}.$$

Multiply both sides by kT.

$$\frac{dT}{dt} = -\frac{kn^2\pi^2}{L^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \quad \rightarrow \quad T_n(t) = \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ over all the eigenvalues.

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\cos\frac{n\pi x}{L}}$$

Use the initial condition $u(x, 0) = f(x)$ to determine A_0 and A_n.

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x) \tag{1}$$

To find A_0, integrate both sides of equation (1) with respect to x from 0 to L.

$$\int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\right) dx = \int_0^L f(x)\, dx$$

Split up the integral on the left and bring the constants in front.

$$A_0 \int_0^L dx + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos\frac{n\pi x}{L}\,dx}_{=\,0} = \int_0^L f(x)\,dx$$

$$A_0 L = \int_0^L f(x)\,dx$$

So then

$$\boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx.}$$

To find A_n, multiply both sides of equation (1) by $\cos(m\pi x/L)$, where m is an integer,

$$A_0 \cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = f(x)\cos\frac{m\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \left(A_0 \cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right) dx = \int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Split up the integral on the left and bring the constants in front.

$$A_0 \underbrace{\int_0^L \cos\frac{m\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Because the cosine functions are orthogonal, the second integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$A_n \int_0^L \cos^2\frac{n\pi x}{L}\,dx = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$A_n\left(\frac{L}{2}\right) = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

So then

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.}$$