Exercise 2.5.10

Using the maximum principles for Laplace’s equation, prove that the solution of Poisson’s equation, $\nabla^2 u = g(x)$, subject to $u = f(x)$ on the boundary, is unique.

Solution

Suppose that there are two solutions, $u$ and $v$, to the Poisson equation in some domain $D$ and its associated boundary condition.

\[
\begin{align*}
\nabla^2 u &= g \quad \text{in } D \\
u &= f \quad \text{on bdy } D
\end{align*}
\]

\[
\begin{align*}
\nabla^2 v &= g \quad \text{in } D \\
v &= f \quad \text{on bdy } D
\end{align*}
\]

Subtract the respective sides of each equation from one another.

\[
\begin{align*}
\nabla^2 u - \nabla^2 v &= g - g \quad \text{in } D \\
u - v &= f - f \quad \text{on bdy } D
\end{align*}
\]

Simplify both sides.

\[
\begin{align*}
\nabla^2 (u - v) &= 0 \quad \text{in } D \\
u - v &= 0 \quad \text{on bdy } D
\end{align*}
\]

Let $w = u - v$.

\[
\begin{align*}
\nabla^2 w &= 0 \quad \text{in } D \\
w &= 0 \quad \text{on bdy } D
\end{align*}
\]

According to the maximum and minimum principles for the Laplace equation, the maximum and minimum values of $w$ occur on the boundary of $D$. If these values are the same, then $w$ has this value inside $D$.

\[
0 \leq w \leq 0 \quad \text{in } D
\]

Therefore, $w = 0$ in $D$, which means the two solutions, $u$ and $v$, are one and the same function.