Exercise 2.5.10

Using the maximum principles for Laplace's equation, prove that the solution of Poisson's equation, $\nabla^2 u = g(\mathbf{x})$, subject to $u = f(\mathbf{x})$ on the boundary, is unique.

Solution

Suppose that there are two solutions, u and v, to the Poisson equation in some domain D and its associated boundary condition.

$$\begin{aligned} \nabla^2 u &= g \quad \text{in } D & \qquad \nabla^2 v &= g \quad \text{in } D \\ u &= f \quad \text{on bdy } D & \qquad v &= f \quad \text{on bdy } D \end{aligned}$$

Subtract the respective sides of each equation from one another.

$$\begin{aligned} \nabla^2 u - \nabla^2 v &= g - g \quad \text{in } D \\ u - v &= f - f \quad \text{on bdy } D \end{aligned}$$

Simplify both sides.

$$\begin{aligned} \nabla^2 (u - v) &= 0 \quad \text{in } D \\ u - v &= 0 \quad \text{on bdy } D \end{aligned}$$

Let $w = u - v$.

$$\begin{aligned} \nabla^2 w &= 0 \quad \text{in } D \\ w &= 0 \quad \text{on bdy } D \end{aligned}$$

According to the maximum and minimum principles for the Laplace equation, the maximum and minimum values of w occur on the boundary of D. If these values are the same, then w has this value inside D.

$$0 \leq w \leq 0 \quad \text{in } D$$

Therefore, $w = 0$ in D, which means the two solutions, u and v, are one and the same function.