Exercise 2.5.12

(a) Using the divergence theorem, determine an alternative expression for $\iint u\nabla^2 u\,dx\,dy\,dz$.

(b) Using part (a), prove that the solution of Laplace's equation $\nabla^2 u = 0$ (with u given on the boundary) is unique.

(c) Modify part (b) if $\nabla u \cdot \hat{\mathbf{n}} = 0$ on the boundary.

(d) Modify part (b) if $\nabla u \cdot \hat{\mathbf{n}} + hu = 0$ on the boundary. Show that Newton's law of cooling corresponds to $h > 0$.

[TYPO: There should be three integral signs here.]

Solution

Note that the following formulas result from the product rule. Subscripts denote partial derivatives.

$$\frac{\partial}{\partial x}(uu_x) = u_x^2 + uu_{xx}$$
$$\frac{\partial}{\partial y}(uu_y) = u_y^2 + uu_{yy}$$
$$\frac{\partial}{\partial z}(uu_z) = u_z^2 + uu_{zz}$$

Part (a)

$$\begin{aligned}
\iiint u\nabla^2 u\,dV &= \iiint u(u_{xx} + u_{yy} + u_{zz})\,dV \\
&= \iiint (uu_{xx} + uu_{yy} + uu_{zz})\,dV \\
&= \iiint \left[\frac{\partial}{\partial x}(uu_x) - u_x^2 + \frac{\partial}{\partial y}(uu_y) - u_y^2 + \frac{\partial}{\partial z}(uu_z) - u_z^2\right] dV \\
&= \iiint \left[\nabla \cdot \langle uu_x, uu_y, uu_z\rangle - (u_x^2 + u_y^2 + u_z^2)\right] dV \\
&= \iiint [\nabla \cdot (u\nabla u) - |\nabla u|^2]\,dV \\
&= \iiint \nabla \cdot (u\nabla u)\,dV - \iiint (\nabla u)^2\,dV
\end{aligned}$$

Apply the divergence theorem to the first volume integral to turn it into a surface integral.

$$\iiint u\nabla^2 u\,dV = \oiint (u\nabla u) \cdot \hat{\mathbf{n}}\,dS - \iiint (\nabla u)^2\,dV \tag{1}$$

Part (b)

Consider the Laplace equation in some domain D with a prescribed Dirichlet boundary condition.

$$\nabla^2 U = 0 \quad \text{in } D$$
$$U = f \quad \text{on bdy } D$$

Suppose there is a second solution to this problem.

$$\nabla^2 V = 0 \quad \text{in } D$$
$$V = f \quad \text{on bdy } D$$

Subtract the respective sides of each equation.

$$\nabla^2 U - \nabla^2 V = 0 \quad \text{in } D$$
$$U - V = f - f \quad \text{on bdy } D$$

Simplify each equation.

$$\nabla^2 (U - V) = 0 \quad \text{in } D$$
$$U - V = 0 \quad \text{on bdy } D$$

Let $W = U - V$.

$$\nabla^2 W = 0 \quad \text{in } D$$
$$W = 0 \quad \text{on bdy } D$$

Multiply both sides of the first equation by W.

$$W\nabla^2 W = 0 \quad \text{in } D$$

Integrate both sides over the volume of D.

$$\iiint_D W\nabla^2 W = 0$$

Use equation (1) here with W instead of u.

$$\underset{\text{bdy } D}{\oiint} (W\nabla W)\cdot \hat{\mathbf{n}}\, dS - \iiint_D (\nabla W)^2\, dV = 0$$

Since $W = 0$ on the boundary of D, this first term vanishes.

$$\underset{\text{bdy } D}{\oiint} (0\nabla W)\cdot \hat{\mathbf{n}}\, dS - \iiint_D (\nabla W)^2\, dV = 0$$

$$-\iiint_D (\nabla W)^2\, dV = 0$$

Multiply both sides by -1.

$$\iiint_D (\nabla W)^2\, dV = 0$$

By the vanishing theorem, the integrand must be zero.

$$(\nabla W)^2 = 0 \quad \text{in } D$$

Expand the left side.

$$W_x^2 + W_y^2 + W_z^2 = 0 \quad \text{in } D$$

Since each term is nonnegative, they must all be zero individually.

$$\left.\begin{array}{lcl} W_x^2 = 0 & \rightarrow & W_x = 0 \\ W_y^2 = 0 & \rightarrow & W_y = 0 \\ W_z^2 = 0 & \rightarrow & W_z = 0 \end{array}\right\} \quad \Rightarrow \quad W = \text{constant} \quad \text{in } D$$

Since $W = 0$ on the boundary of D, this constant must be zero because W is continuous.

$$W = 0 \quad \text{in } D$$

This means the two solutions, U and V, are one and the same function. Therefore, the solution to the Laplace equation with a Dirichlet boundary condition is unique.

Part (c)

Consider the Laplace equation in some domain D with a homogeneous Neumann boundary condition.

$$\begin{aligned} \nabla^2 U &= 0 \quad \text{in } D \\ \nabla U \cdot \hat{\mathbf{n}} &= 0 \quad \text{on bdy } D \end{aligned}$$

Suppose there is a second solution to this problem.

$$\begin{aligned} \nabla^2 V &= 0 \quad \text{in } D \\ \nabla V \cdot \hat{\mathbf{n}} &= 0 \quad \text{on bdy } D \end{aligned}$$

Subtract the respective sides of each equation.

$$\begin{aligned} \nabla^2 U - \nabla^2 V &= 0 \quad \text{in } D \\ \nabla U \cdot \hat{\mathbf{n}} - \nabla V \cdot \hat{\mathbf{n}} &= 0 \quad \text{on bdy } D \end{aligned}$$

Simplify each equation.

$$\begin{aligned} \nabla^2 (U - V) &= 0 \quad \text{in } D \\ \nabla (U - V) \cdot \hat{\mathbf{n}} &= 0 \quad \text{on bdy } D \end{aligned}$$

Let $W = U - V$.

$$\begin{aligned} \nabla^2 W &= 0 \quad \text{in } D \\ \nabla W \cdot \hat{\mathbf{n}} &= 0 \quad \text{on bdy } D \end{aligned}$$

Multiply both sides of the first equation by W.

$$W \nabla^2 W = 0 \quad \text{in } D$$

Integrate both sides over the volume of D.

$$\iiint_D W\nabla^2 W = 0$$

Use equation (1) here with W instead of u.

$$\oiint_{\text{bdy } D} (W\nabla W)\cdot\hat{\mathbf{n}}\,dS - \iiint_D (\nabla W)^2\,dV = 0$$

Since $\nabla W\cdot\hat{\mathbf{n}} = 0$ on the boundary of D, this first term vanishes.

$$\oiint_{\text{bdy } D} (W\cdot 0)\,dS - \iiint_D (\nabla W)^2\,dV = 0$$

$$-\iiint_D (\nabla W)^2\,dV = 0$$

Multiply both sides by -1.

$$\iiint_D (\nabla W)^2\,dV = 0$$

By the vanishing theorem, the integrand must be zero.

$$(\nabla W)^2 = 0 \quad \text{in } D$$

Expand the left side.

$$W_x^2 + W_y^2 + W_z^2 = 0 \quad \text{in } D$$

Since each term is nonnegative, they must all be zero individually.

$$\left.\begin{array}{lcl} W_x^2 = 0 & \rightarrow & W_x = 0 \\ W_y^2 = 0 & \rightarrow & W_y = 0 \\ W_z^2 = 0 & \rightarrow & W_z = 0 \end{array}\right\} \quad \Rightarrow \quad W = \text{constant} \quad \text{in } D$$

Therefore, the solution to the Laplace equation with a homogeneous Neumann boundary condition is unique to within an additive constant.

Part (d)

Consider the Laplace equation in some domain D with a homogeneous Robin boundary condition.

$$\begin{aligned} \nabla^2 U &= 0 \quad \text{in } D \\ \nabla U \cdot \hat{\mathbf{n}} + hU &= 0 \quad \text{on bdy } D \end{aligned}$$

Suppose there is a second solution to this problem.

$$\begin{aligned} \nabla^2 V &= 0 \quad \text{in } D \\ \nabla V \cdot \hat{\mathbf{n}} + hV &= 0 \quad \text{on bdy } D \end{aligned}$$

Subtract the respective sides of each equation.

$$\begin{aligned} \nabla^2 U - \nabla^2 V &= 0 \quad \text{in } D \\ \nabla U \cdot \hat{\mathbf{n}} - \nabla V \cdot \hat{\mathbf{n}} + hU - hV &= 0 \quad \text{on bdy } D \end{aligned}$$

Simplify each equation.

$$\begin{aligned} \nabla^2 (U - V) &= 0 \quad \text{in } D \\ \nabla (U - V) \cdot \hat{\mathbf{n}} + h(U - V) &= 0 \quad \text{on bdy } D \end{aligned}$$

Let $W = U - V$.

$$\begin{aligned} \nabla^2 W &= 0 \quad \text{in } D \\ \nabla W \cdot \hat{\mathbf{n}} + hW &= 0 \quad \text{on bdy } D \end{aligned}$$

Multiply both sides of the first equation by W.

$$W \nabla^2 W = 0 \quad \text{in } D$$

Integrate both sides over the volume of D.

$$\iiint_D W \nabla^2 W = 0$$

Use equation (1) here with W instead of u.

$$\oiint_{\text{bdy } D} (W \nabla W) \cdot \hat{\mathbf{n}} \, dS - \iiint_D (\nabla W)^2 \, dV = 0$$

Use the fact that $\nabla W \cdot \hat{\mathbf{n}} = -hW$ on the boundary of D.

$$\oiint_{\text{bdy } D} [W(-hW)] \, dS - \iiint_D (\nabla W)^2 \, dV = 0$$

$$-\oiint_{\text{bdy } D} hW^2 \, dS - \iiint_D (\nabla W)^2 \, dV = 0$$

Multiply both sides by -1.

$$\oiint_{\text{bdy } D} hW^2\,dS + \iiint_D (\nabla W)^2\,dV = 0$$

Provided that $h > 0$, each integrand is nonnegative, which means each of the integrals is nonnegative. Each term must be zero individually for the equation to be satisfied.

$$\begin{cases} \displaystyle\oiint_{\text{bdy } D} hW^2\,dS = 0 \\ \displaystyle\iiint_D (\nabla W)^2\,dV = 0 \end{cases}$$

By the vanishing theorem, each integrand must be zero.

$$\begin{cases} hW^2 = 0 & \text{on bdy } D \\ (\nabla W)^2 = 0 & \text{in } D \end{cases}$$

$$\begin{cases} W = 0 & \text{on bdy } D \\ W_x^2 + W_y^2 + W_z^2 = 0 & \text{in } D \end{cases}$$

Since each term is nonnegative in the second equation, they must all be zero individually.

$$\left.\begin{aligned} W_x^2 = 0 \quad &\rightarrow \quad W_x = 0 \\ W_y^2 = 0 \quad &\rightarrow \quad W_y = 0 \\ W_z^2 = 0 \quad &\rightarrow \quad W_z = 0 \end{aligned}\right\} \quad \Rightarrow \quad W = \text{constant} \quad \text{in } D$$

Since $W = 0$ on the boundary of D, this constant must be zero because W is continuous.

$$W = 0 \quad \text{in } D$$

This means the two solutions, U and V, are one and the same function. Therefore, the solution to the Laplace equation with a homogeneous Robin boundary condition is unique if $h > 0$. Writing the boundary condition as

$$\underbrace{-\nabla u \cdot \hat{\mathbf{n}}}_{\text{heat flux}} = hu,$$

we see that the heat flux out of D (in the direction of $\hat{\mathbf{n}}$, the outward unit normal vector) is proportional to the temperature on the boundary of D. If $h > 0$, this outward flow of heat from the boundary results in cooling. If $h < 0$, then heat will flow inward, resulting in heating.