Exercise 2.5.16

Consider Laplace's equation inside a rectangle $0 \leq x \leq L$, $0 \leq y \leq H$, with the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(L,y) = g(y), \quad \frac{\partial u}{\partial y}(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,H) = f(x).$$

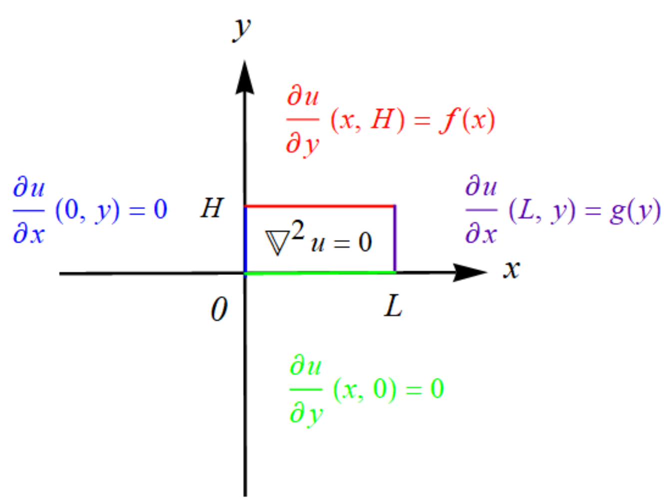
(a) What is the solvability condition and its physical interpretation?

(b) Show that $u(x,y) = A(x^2 - y^2)$ is a solution if $f(x)$ and $g(y)$ are constants [under the conditions of part (a)].

(c) Under the conditions of part (a), solve the general case [nonconstant $f(x)$ and $g(y)$]. [*Hints:* Use part (b) and the fact that $f(x) = f_{\text{av}} + [f(x) - f_{\text{av}}]$, where $f_{\text{av}} = \frac{1}{L}\int_0^L f(x)\,dx$.]

Solution

Part (a)



$$\nabla^2 u = 0$$

To obtain the solvability condition, integrate both sides of the Laplace equation over the rectangular domain.

$$\iint_R \nabla^2 u \, dA = 0$$

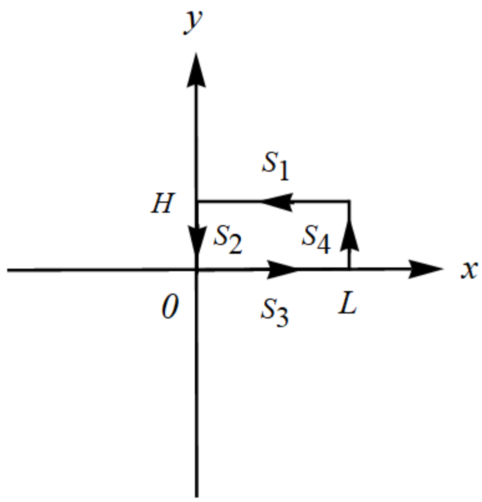
Rewrite the integrand.

$$\iint_R \nabla \cdot \nabla u \, dA = 0$$

Apply the two-dimensional divergence theorem to change this double integral into a counterclockwise closed loop integral around the boundary.

$$\oint_S \nabla u \cdot \hat{\mathbf{n}} \, ds = 0$$

Here $\hat{\mathbf{n}}$ is the outward unit vector normal to the boundary.



This boundary S can be split into four segments, S_1, S_2, S_3, and S_4. The outward normal unit vector to each of them is $\hat{\mathbf{n}} = \hat{\mathbf{y}}$, $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$, $\hat{\mathbf{n}} = -\hat{\mathbf{y}}$, and $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, respectively.

$$\int_{S_1} \nabla u \cdot \hat{\mathbf{y}}\, ds + \int_{S_2} \nabla u \cdot (-\hat{\mathbf{x}})\, ds + \int_{S_3} \nabla u \cdot (-\hat{\mathbf{y}})\, ds + \int_{S_4} \nabla u \cdot \hat{\mathbf{x}}\, ds = 0$$

Evaluate the dot products.

$$\int_{S_1} \frac{\partial u}{\partial y}\, ds + \int_{S_2} \left(-\frac{\partial u}{\partial x}\right) ds + \int_{S_3} \left(-\frac{\partial u}{\partial y}\right) ds + \int_{S_4} \frac{\partial u}{\partial x}\, ds = 0$$

Bring the minus signs in front.

$$\int_{S_1} \frac{\partial u}{\partial y}\, ds - \int_{S_2} \frac{\partial u}{\partial x}\, ds - \int_{S_3} \frac{\partial u}{\partial y}\, ds + \int_{S_4} \frac{\partial u}{\partial x}\, ds = 0$$

Along S_1, u_y is evaluated at $y = H$; along S_2, u_x is evaluated at $x = 0$; along S_3, u_y is evaluated at $y = 0$; and along S_4, u_x is evaluated at $x = L$.

$$\int_{S_1} \left.\frac{\partial u}{\partial y}\right|_{y=H} ds - \int_{S_2} \left.\frac{\partial u}{\partial x}\right|_{x=0} ds - \int_{S_3} \left.\frac{\partial u}{\partial y}\right|_{y=0} ds + \int_{S_4} \left.\frac{\partial u}{\partial x}\right|_{x=L} ds = 0$$

The differential of arc length ds is always positive regardless of whether the path around the boundary is clockwise or counterclockwise. So don't mind the orientation when parameterizing the integration paths.

$$\int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=H} dx - \int_0^H \left.\frac{\partial u}{\partial x}\right|_{x=0} dy - \int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=0} dx + \int_0^H \left.\frac{\partial u}{\partial x}\right|_{x=L} dy = 0$$

Substitute the prescribed boundary conditions, $u_x(0, y) = 0$ and $u_x(L, y) = g(y)$ and $u_y(x, 0) = 0$ and $u_y(x, H) = f(x)$.

$$\int_0^L f(x)\, dx - \int_0^H 0\, dy - \int_0^L 0\, dx + \int_0^H g(y)\, dy = 0$$

$$\int_0^L f(x)\, dx + \int_0^H g(y)\, dy = 0$$

This is the solvability condition, which must be satisfied for there to be a steady state. Assuming that u represents temperature, it means that the net heat flux entering the rectangle must be zero; otherwise, the temperature would rise or drop indefinitely, and there wouldn't be a steady state. f and g are not arbitrary functions.

Part (b)

Suppose that f and g are constants. Then the boundary conditions are

$$\begin{aligned} \frac{\partial u}{\partial y}(x,0) &= 0 & \qquad \frac{\partial u}{\partial y}(x,H) &= f(x) = C_1 \\ \frac{\partial u}{\partial y}(0,y) &= 0 & \qquad \frac{\partial u}{\partial x}(L,y) &= g(y) = C_2. \end{aligned}$$

Observe that the derivatives increase in a linear fashion.

$$\frac{\partial u}{\partial y} = \frac{C_1}{H}y$$
$$\frac{\partial u}{\partial x} = \frac{C_2}{L}x$$

The differential of a two-dimensional function $u = u(x, y)$ is defined to be

$$\begin{aligned} du &= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \\ &= \frac{C_2}{L}x\,dx + \frac{C_1}{H}y\,dy. \end{aligned}$$

Integrate both sides to get u.

$$\begin{aligned} u(x,y) &= \int^x \frac{C_2}{L}x\,dx + \int^y \frac{C_1}{H}y\,dy + C \\ &= \frac{C_2}{2L}x^2 + \frac{C_1}{2H}y^2 + C \end{aligned}$$

Apply the solvability condition to determine one of the constants.

$$\int_0^L f(x)\,dx + \int_0^H g(y)\,dy = 0$$
$$\int_0^L C_1\,dx + \int_0^H C_2\,dy = 0$$
$$C_1L + C_2H = 0$$
$$C_1 = -\frac{H}{L}C_2$$

As a result,

$$\begin{aligned} u(x,y) &= \frac{C_2}{2L}x^2 - \frac{H}{L}\frac{C_2}{2H}y^2 + C \\ &= \frac{C_2}{2L}(x^2 - y^2) + C. \end{aligned}$$

Use a new arbitrary constant A for $C_2/(2L)$ and set $C = 0$ to obtain the desired solution.

$$u(x,y) = A(x^2 - y^2)$$

Part (c)

Here the Laplace equation will be solved in the rectangle ($0 \leq x \leq L$, $0 \leq y \leq H$) that is subject to Neumann boundary conditions on all sides. The solvability condition is assumed to be satisfied.

$$\begin{aligned}
&\nabla^2 u = 0, \qquad 0 < x < L,\ 0 < y < H \\
&u_x(0, y) = 0 \\
&u_x(L, y) = g(y) \\
&u_y(x, 0) = 0 \\
&u_y(x, H) = f(x)
\end{aligned}$$

Use the fact that the PDE is linear to split up the problem to make it so that there's only one inhomogeneous boundary condition in each.

$$\begin{aligned}
&\nabla^2 v = 0, \qquad 0 < x < L,\ 0 < y < H &\qquad& \nabla^2 w = 0, \qquad 0 < x < L,\ 0 < y < H \\
&v_x(0, y) = 0 && w_x(0, y) = 0 \\
&v_x(L, y) = 0 && w_x(L, y) = g(y) \\
&v_y(x, 0) = 0 && w_y(x, 0) = 0 \\
&v_y(x, H) = f(x) && w_y(x, H) = 0
\end{aligned}$$

The final solution will be the sum of v and w: $u(x, y) = v(x, y) + w(x, y)$. Solve the boundary value problem for v first by using the method of separation of variables. Assume a product solution of the form $v(x, y) = X(x)Y(y)$ and plug it into the PDE

$$\frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0 \quad \rightarrow \quad X''Y + XY'' = 0 \quad \rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned}
v_x(0, y) = 0 \quad &\rightarrow \quad X'(0)Y(y) = 0 \quad &\rightarrow \quad X'(0) = 0 \\
v_x(L, y) = 0 \quad &\rightarrow \quad X'(L)Y(y) = 0 \quad &\rightarrow \quad X'(L) = 0 \\
v_y(x, 0) = 0 \quad &\rightarrow \quad X(x)Y'(0) = 0 \quad &\rightarrow \quad Y'(0) = 0
\end{aligned}$$

Separate variables in the PDE.

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{aligned}
\frac{X''}{X} &= \lambda \\
-\frac{Y''}{Y} &= \lambda
\end{aligned}\right\}$$

Values of λ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Note that it doesn't matter what side the minus sign is on as long as all eigenvalues are considered.

Solve the ODE for X.

$$X'' = \lambda X$$

Check to see if there are positive eigenvalues: $\lambda = \mu^2$.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \mu x + C_4 \sinh \mu x$$

Differentiate both sides with respect to x.

$$X'(x) = \mu(C_3 \sinh \mu x + C_4 \cosh \mu x)$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X'(0) &= \mu(C_4) = 0 \\ X'(L) &= \mu(C_3 \sinh \mu L + C_4 \cosh \mu L) = 0 \end{aligned}$$

Since $C_4 = 0$, this second equation reduces to $C_3 \mu \sinh \mu L = 0$. No nonzero value of μ can satisfy this equation, so $C_3 = 0$.

$$X(x) = 0$$

The trivial solution is obtained, so there are no positive eigenvalues. Check to see if zero is an eigenvalue.

$$X'' = 0$$

The general solution is a straight line.

$$X(x) = C_5 x + C_6$$

Differentiate it with respect to x.

$$X'(x) = C_5$$

Apply the boundary conditions to determine C_5.

$$\begin{aligned} X'(0) &= C_5 = 0 \\ X'(L) &= C_5 = 0 \end{aligned}$$

C_6 remains arbitrary.

$$X(x) = C_6$$

This is not the trivial solution, so zero is an eigenvalue. Using $\lambda = 0$, solve the ODE for Y now.

$$Y'' = 0$$

The general solution is a straight line.

$$Y(y) = Dy + E$$

Set $D = 0$ so that $Y'(0) = 0$ is satisfied.

$$Y(y) = E$$

Now check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$X'' = -\gamma^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_7 \cos \gamma x + C_8 \sin \gamma x$$

Differentiate it with respect to x.

$$X'(x) = \gamma(-C_7 \sin \gamma x + C_8 \cos \gamma x)$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} X'(0) &= \gamma(C_8) = 0 \\ X'(L) &= \gamma(-C_7 \sin \gamma L + C_8 \cos \gamma L) = 0 \end{aligned}$$

Since $C_8 = 0$, this second equation reduces to $-C_7 \gamma \sin \gamma L = 0$. To avoid the trivial solution, we insist that $C_7 \neq 0$.

$$\begin{aligned} \sin \gamma L &= 0 \\ \gamma L &= n\pi, \quad n = 1, 2, \ldots \\ \gamma &= \frac{n\pi}{L} \end{aligned}$$

There are negative eigenvalues $\lambda = -\left(\frac{n\pi}{L}\right)^2$, and the eigenfunctions associated with them are

$$X(x) = C_7 \cos \gamma x \quad \rightarrow \quad X_n(x) = \cos \frac{n\pi x}{L}.$$

Note that n is taken over the positive integers only because $n = 0$ leads to the zero eigenvalue, and negative integers lead to redundant values for λ. Using $\lambda = -\frac{n^2\pi^2}{L^2}$, solve the ODE for Y now.

$$Y'' = \frac{n^2\pi^2}{L^2} Y$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = D_1 \cosh \frac{n\pi y}{L} + E_1 \sinh \frac{n\pi y}{L}$$

Differentiate it with respect to y.

$$Y'(y) = \frac{n\pi}{L} \left(D_1 \sinh \frac{n\pi y}{L} + E_1 \cosh \frac{n\pi y}{L} \right)$$

Apply the boundary condition for Y to determine one of the constants.

$$Y'(0) = \frac{n\pi}{L}(E_1) = 0 \quad \rightarrow \quad E_1 = 0$$

The Y-eigenfunction is then

$$Y(y) = D_1 \cosh \frac{n\pi y}{L}.$$

According to the principle of superposition, the general solution to the PDE for v is a linear combination of the eigenfunctions $v = X(x)Y(y)$ over all the eigenvalues.

$$v(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi y}{L} \cos \frac{n\pi x}{L}$$

Differentiate it with respect to y.

$$\frac{\partial v}{\partial y} = \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh \frac{n\pi y}{L} \cos \frac{n\pi x}{L}$$

Apply the final boundary condition to determine the coefficients A_n.

$$\frac{\partial v}{\partial y}(x, H) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh \frac{n\pi H}{L} \cos \frac{n\pi x}{L} = f(x)$$

Multiply both sides by $\cos \frac{p\pi x}{L}$, where p is an integer.

$$\sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh \frac{n\pi H}{L} \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L} = f(x) \cos \frac{p\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh \frac{n\pi H}{L} \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}\, dx = \int_0^L f(x) \cos \frac{p\pi x}{L}\, dx$$

Split up the integrals and bring the constants in front.

$$\sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh \frac{n\pi H}{L} \int_0^L \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}\, dx = \int_0^L f(x) \cos \frac{p\pi x}{L}\, dx$$

Because the cosine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero value.

$$A_n \frac{n\pi}{L} \sinh \frac{n\pi H}{L} \int_0^L \cos^2 \frac{n\pi x}{L}\, dx = \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

Evaluate the integral.

$$A_n \frac{n\pi}{L} \sinh \frac{n\pi H}{L} \left(\frac{L}{2}\right) = \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

Therefore,

$$\boxed{A_n = \frac{2}{n\pi \sinh \frac{n\pi H}{L}} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx.}$$

Now solve the boundary value problem for w.

$$\begin{aligned}
&\nabla^2 w = 0, \qquad 0 < x < L,\ 0 < y < H \\
&w_x(0, y) = 0 \\
&w_x(L, y) = g(y) \\
&w_y(x, 0) = 0 \\
&w_y(x, H) = 0
\end{aligned}$$

Use the method of separation of variables again and assume a product solution of the form $w(x, y) = X(x)Y(y)$. Plug it into the PDE

$$\frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0 \quad \rightarrow \quad X''Y + XY'' = 0 \quad \rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

and the homogeneous boundary conditions.

$$\begin{array}{ccccc}
w_x(0, y) = 0 & \rightarrow & X'(0)Y(y) = 0 & \rightarrow & X'(0) = 0 \\
w_y(x, 0) = 0 & \rightarrow & X(x)Y'(0) = 0 & \rightarrow & Y'(0) = 0 \\
w_y(x, H) = 0 & \rightarrow & X(x)Y'(H) = 0 & \rightarrow & Y'(H) = 0
\end{array}$$

This is the same problem as before for v except that the X- and Y- eigenfunctions are switched and L is replaced with H. The general solution for w is then

$$w(x, y) = B_0 + \sum_{n=1}^{\infty} B_n \cosh \frac{n\pi x}{H} \cos \frac{n\pi y}{H}.$$

Differentiate it with respect to x.

$$\frac{\partial w}{\partial x} = \sum_{n=1}^{\infty} B_n \frac{n\pi}{H} \sinh \frac{n\pi x}{H} \cos \frac{n\pi y}{H}$$

Apply the final boundary condition to determine the coefficients B_n.

$$\frac{\partial w}{\partial x}(L, y) = \sum_{n=1}^{\infty} B_n \frac{n\pi}{H} \sinh \frac{n\pi L}{H} \cos \frac{n\pi y}{H} = g(y)$$

Multiply both sides by $\cos \frac{p\pi y}{H}$, where p is an integer.

$$\sum_{n=1}^{\infty} B_n \frac{n\pi}{H} \sinh \frac{n\pi L}{H} \cos \frac{n\pi y}{H} \cos \frac{p\pi y}{H} = g(y) \cos \frac{p\pi y}{H}$$

Integrate both sides with respect to y from 0 to H.

$$\int_0^H \sum_{n=1}^{\infty} B_n \frac{n\pi}{H} \sinh \frac{n\pi L}{H} \cos \frac{n\pi y}{H} \cos \frac{p\pi y}{H}\, dy = \int_0^H g(y) \cos \frac{p\pi y}{H}\, dy$$

Split up the integral on the left and bring the constants in front.

$$\sum_{n=1}^{\infty} B_n \frac{n\pi}{H} \sinh \frac{n\pi L}{H} \int_0^H \cos \frac{n\pi y}{H} \cos \frac{p\pi y}{H}\, dy = \int_0^H g(y) \cos \frac{p\pi y}{H}\, dy$$

Because the cosine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero value.

$$B_n \frac{n\pi}{H} \sinh \frac{n\pi L}{H} \int_0^H \cos^2 \frac{n\pi y}{H}\, dy = \int_0^H g(y) \cos \frac{n\pi y}{H}\, dy$$

Evaluate the integral.

$$B_n \frac{n\pi}{H} \sinh \frac{n\pi L}{H} \left(\frac{H}{2}\right) = \int_0^H g(y) \cos \frac{n\pi y}{H}\, dy$$

Therefore,

$$\boxed{B_n = \frac{2}{n\pi \sinh \frac{n\pi L}{H}} \int_0^H g(y) \cos \frac{n\pi y}{H}\, dy,}$$

and since $u(x, y) = v(x, y) + w(x, y)$,

$$\boxed{u(x, y) = (A_0 + B_0) + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi y}{L} \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \cosh \frac{n\pi x}{H} \cos \frac{n\pi y}{H},}$$

where $A_0 + B_0$ remains arbitrary.