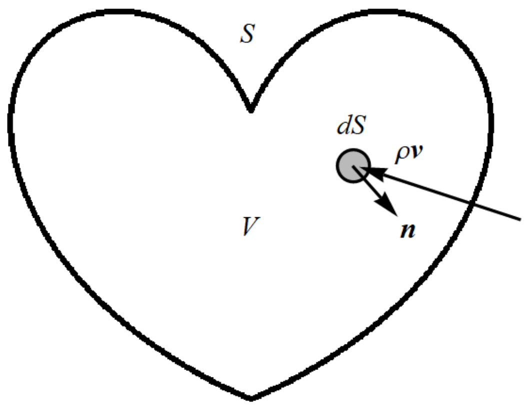
Exercise 2.5.17

Show that the mass density $\rho(x,t)$ satisfies $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$ due to conservation of mass.

Solution

Consider a fixed three-dimensional arbitrary control volume with constant volume V and surface area S through which a fluid flows. Let $\mathbf{n}$ be the outward unit vector normal to the boundary.



Make a mass balance over this control volume.

$$\begin{bmatrix}\text{rate of mass increase}\\ \text{with respect to time}\end{bmatrix} = \begin{bmatrix}\text{rate of mass}\\ \text{in}\end{bmatrix} - \begin{bmatrix}\text{rate of mass}\\ \text{out}\end{bmatrix} + \begin{bmatrix}\text{rate of mass}\\ \text{accumulation}\end{bmatrix}$$

A fluid with density ρ and velocity $\mathbf{v}$ carries a momentum $\rho\mathbf{v}$ per unit volume. Integrating $\rho\mathbf{v}$ over the surface of the control volume gives the rate that mass flows in minus the rate that mass flows out. By the law of conservation of mass, mass is neither created nor destroyed, so the rate of accumulation is zero.

$$\frac{dm}{dt} = \oiint_S \rho\mathbf{v} \cdot d\mathbf{S}$$

Mass that's being transported into the control volume is antiparallel to $\mathbf{n}$, making "rate of mass in" negative, and mass that's being transported out of the control volume is parallel to $\mathbf{n}$, making "rate of mass out" positive. In other words, the dot product yields a minus sign.

$$\frac{dm}{dt} = \oiint_S (-\rho\mathbf{v}) \cdot (\mathbf{n}\, dS)$$

The mass within the control volume is obtained by integrating the density ρ over the volume V.

$$\frac{d}{dt} \iiint_V \rho\, dV = -\oiint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS$$

V is constant, so the time derivative can be brought inside the integral.

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = -\oiint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS$$

Apply the divergence theorem to turn the surface integral into a volume integral.

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = -\iiint_V \nabla \cdot \rho\mathbf{v}\, dV$$

Bring both integrals to the left side.

$$\iiint_V \frac{\partial \rho}{\partial t}\,dV + \iiint_V \nabla \cdot \rho \mathbf{v}\,dV = 0$$

Combine them.

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v}\right) dV = 0$$

Since the choice of the control volume is arbitrary, the volume integral may be removed.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

This is the continuity equation.